

Topology

M.Sc. Sem I Mathematics

Set - Set is collection of well defined objects.

Subjects :- $A = \{ \text{---} : \text{---} \}$
 $\swarrow$ object's definition
 $\searrow$ object

Subset :- let A be any set and B be another set (if may be similar as A) if for any $x \in B \Rightarrow x \in A$ then we say B is subset of A and we denote it as $B \subseteq A$ (B is subset of A)
 or $A \supseteq B$ (A is superset of A)

* Power Set - $A = \{1, 2\}$

$$A_1 = \{1\}, A_2 = \{2\}$$

$$A_3 = \{1, 2\}, A_4 = \{\}$$

Power set of $A = \{A_1, A_2, A_3, A_4\}$

$$= \{\{\}, \{1\}, \{2\}, \{1, 2\}\}$$

power set - collection of all subsets of any set A is known as power set of A .

$P[A]$ represents power set of A .

Empty set / Null set / void set $\Rightarrow$ Set without any elements $\{\}$,
 $\emptyset$ are used to represent empty set.

Theorem:- Prove that empty set is subset of every set.

proof:- let A be any set and ϕ is empty set then we have to show that ϕ is subset of A . i.e. $\phi \subseteq A$

$$[x \in \phi] \Rightarrow x \in A \quad \text{--- } x.$$

here $\nexists$ any $x \in A$

such that $x \in \phi$

$$\text{i.e. } x \notin A \Rightarrow x \notin \phi$$

Thus ϕ is subset of A

$$A = \{1, 2, 3, 4, 5, 6\}$$

$$B = \{1, 2\}$$

$$C = \{1, 2, 7\}$$

~~proof~~

$$7 \notin A \text{ but } 7 \in C$$

Finite Set $\Rightarrow$ A set is called finite set if it has finite no. of elements.

ex.

$$A = \{1, 2, 3, 4, 5\} \text{ is finite set}$$

$$B = \{a, b\} \text{ is a finite set.}$$

$$(\phi \text{ is finite set})$$

Infinite Set - A set is called infinite set if it has infi

$$\text{Function } \Rightarrow f: \underset{\text{set}}{A} \rightarrow \underset{\text{set}}{B}$$

$$R = \{(x, y) \in \mathbb{N} \times \mathbb{N} : x = y\}$$

$$R = \{(1, 1), (2, 2), (3, 3), \dots\}$$

or

$$R(1) = \{1\}$$

$$R(2) = \{2\}$$

which of the following is a function from

$$A = \{1, 2, 3\} \text{ to } B = \{5, 6\}$$

- (a) $R_1 = \{(1, 1), (1, 2), (1, 3)\}$ Not Relation & finite
(b) $R_2 = \{(1, 2), (2, 3), (3, 4)\}$ not Re & fin.
(c) $R_3 = \{(1, 5), (1, 6), (2, 5)\}$ is Rel. & not f
(d) $R_4 = \{(1, 5), (2, 5)\}$ Relation & not fin
(e) $R_5 = \{(1, 5), (2, 6)\}$ Relation & not finite
(f) $R_6 = \{(1, 5), (2, 5), (3, 5)\}$ function & Relation
(g) $R_7 = \{(1, 6), (2, 5), (3, 5)\}$ function & Relation

$$R = \{(x, y) \in A \times B : \text{Def}^n \text{ for Relation}\}$$

here B is Co-Domain of relation & $D(f)$ is domain of relation.

if R relates elements from $D(R)$ to a unique element of co-domain. Then R is known as function and written as

$$R: D(f) \rightarrow B$$

$f: D(f) \subseteq A \rightarrow B$ be a function
then f is s.t.b one-one i.e.

into

$$f(x) = f(y)$$

$$\Rightarrow x = y$$

* Cardinal Numbers - A numbers which is used to represent potential or number of elements in any set.

* Transfinite Cardinals - A numbers which is used to represent cardinality of any infinite set is known as transfinite Cardinals.

eg. $\aleph_0$ - null

$\mathbb{C}$ - contin.

Similar Set - (equipotential set) - Let A & B

$[A \sim B]$

A & B are s.t.b similar if $\exists$ a bijection them i.e. $\rightarrow$ a function $f: A \rightarrow B$ so that f is one-one & onto

* Cardinality of any set is total number of elements in that set -

If A is any set then

$$|A| = \text{cardinality of } A$$

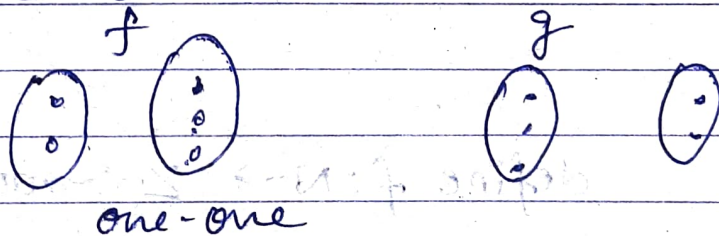
eg: $A = \{1, 2\}, \quad |\emptyset| = 0$

$$|A| = 2$$

$$|N| = \aleph_0$$

Q. if $|A| \leq |B|$ & $f: A \rightarrow B$ & $g: B \rightarrow A$ then

- (a) f & g can be one one
- (b) f & g can be onto
- (c) f can be one one
- (d) g can be one one



* Countable infinite set - Any set which is similar to N (Set of natural no.) is called countable infinite set.

* Countable set - A set which is either finite or countable infinite is known as countable set.

$\mathbb{N}$ = natural no. $\mathbb{R}$ = Real no. $\mathbb{Q}$ = Rational
 $\mathbb{Z}$ = Integers, $\mathbb{C}$ = Complex no.

PAGE NO.:

DATE:

* Uncountable set - A ^{set} which is not countable is called uncountable set.

Q Prove that $\mathbb{Z}$ & $\mathbb{N}$ are similar.

$$\mathbb{N} = \{1, 2, 3, \dots\} \quad \text{Natural no.}$$
$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

prove —

$\mathbb{Z}$ is set of integers & $\mathbb{N}$ is set of natural numbers.

similar Set - Set A & B be any set then A & B are st.b similar

if $\Rightarrow$

define $f: \mathbb{N} \rightarrow \mathbb{Z}$ such that

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ \frac{-(n-1)}{2} & \text{if } n \text{ is odd} \end{cases}$$

(i) f is one-one - let $\exists x$ & y such that $f(x) = f(y)$

Case I - If $f(x)$ is positive.

$$\Rightarrow \frac{x}{2} = \frac{y}{2}$$

$$\Rightarrow x = y$$

Case II if $f(x)$ is negative

$$\frac{-(x-1)}{2} = \frac{-(y-1)}{2}$$

$$\Rightarrow x-1 = y-1$$

$$\Rightarrow x = y$$

Case III if $f(x) = 0$

$$\frac{-(x-1)}{2} = \frac{-(y-1)}{2} = 0$$

$$\Rightarrow x = y$$

Thus $f(x) = f(y)$

$$\Rightarrow x = y$$

therefore f is one-one

(ii) f is onto:

here $1 \in \mathbb{N}$ so that

$$f(1) = 0$$

let $n \in \mathbb{Z}$ and $n > 0$

$$n = \frac{2n}{2}$$

$$= \frac{2n}{2}$$

$$= f(2n)$$

Now $n > 0$ & $n \in \mathbb{Z}$

$$\Rightarrow n \in \mathbb{N}$$

$$\Rightarrow 2n \in \mathbb{N}$$

again let $\bar{m} \in \mathbb{Z}$ & $\bar{m} \in \mathbb{Z}$
 $\Rightarrow \exists m \in \mathbb{N}$ such that

$$= \frac{-2m+1-1}{2} \leq m = -m$$

$$= \frac{-(2m+1)-1}{2}$$

$$= f(2m+1)$$

$$\text{have } m \in \mathbb{N} \Rightarrow 2m \in \mathbb{N}$$

$$\Rightarrow 2m+1 \in \mathbb{N}$$

Thy f is one-one & onto

If $\mathbb{Z} \sim \mathbb{N}$ are similar

$$\therefore \mathbb{Z} \sim \mathbb{N}$$

$$Q. \mathbb{N} \times \mathbb{N} \sim \mathbb{N} [X_0^2 = X_0]$$

$$Q. \mathbb{N} \times \mathbb{N} \dots \times \mathbb{N} \sim \mathbb{N} [X_0^n = X_0]$$

} algebra of cardinal number

theorem - Prove that countable union of countable set is countable.

prove - let $\{A_i : i \in \mathbb{N}\}$ be the collection of countable infinite set and $\mathbb{N}$ is countable.

$$\text{Define } A = \bigcup_{i=1}^{\infty} A_i \quad (\mathbb{N} = \mathbb{N})$$

[countable union of countable set]

Then we have to show that A is countable.

Since A_i is countable

$$A_i \sim \mathbb{N}$$

$$A_i = \{a_{i1}, a_{i2}, a_{i3}, \dots\} \quad \forall a_i \in \mathbb{N}$$

$$A = \{a_{11}, a_{12}, a_{13}, \dots$$

$$a_{21}, a_{22}, a_{23}, \dots$$

$A_{n1}, A_{n2}, \dots$
 $\vdots$
 $\dots$

Now we get an arrangement of the element of A in following order.

$a_{11}, a_{21}, a_{12}, a_{31}, a_{22}, a_{13},$
 $\dots$

which is in natural order

Thus A is countable.

$\mathbb{N} \times \mathbb{N}$ is countable.

$$\rightarrow \mathbb{N} \times \mathbb{N} = \{ (m, n) : m, n \in \mathbb{N} \}$$

$$= \{ (1, 1), (1, 2), (1, 3), \dots$$

$$(2, 1), (2, 2), (2, 3), \dots \rightarrow$$

$$(3, 1), (3, 2), \dots$$

Thus it is countable

Q. prove that $\aleph_0^2 = \aleph_0$

$$\downarrow$$

$$|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$$

* Every subset of countable set is countable
 proof: let A be any countable set.

and B is subset A

Then we have to show that -

B is countable.

case I If A is finite.

Then B is also finite.

$\Rightarrow B$ is countable.

Case II if A is inf countable infinite.

$$A = \{a_1, a_2, a_3, \dots\}$$

If B is finite then it is countable.

If B is infinite:

We check element of A in their natural order and i is the smallest natural no.

for which $a_i \in B$. So we remove

it as b_1 , i.e. $a_i = b_1$

we continue in this way

and we get $b_2, b_3, b_4, \dots$

all are the elements of B

$$\text{So } B = \{b_1, b_2, b_3, \dots\}$$

If $x \in B \Rightarrow x \in A$

$\Rightarrow \exists j \in \mathbb{N}$ such that

$$x = a_j$$

$$\Rightarrow \exists k \in \mathbb{N} \text{ such that } x = a_j = b_k$$

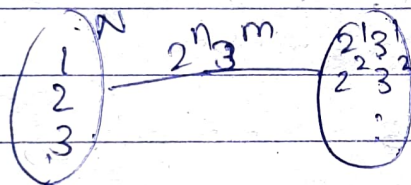
Then B is countable.

Theorem:- Show that $\mathbb{N} \times \mathbb{N}$ is countable infinite set.

proof: let $B = \mathbb{N} \times \mathbb{N}$

$$\text{then } B = \{(n, m) | n \in \mathbb{N}, m \in \mathbb{N}\}$$

Define a mapping $f: \mathbb{N} \rightarrow B$
by $f(n, m) = 2^n 3^m$



Claim:- f is an injective mapping

verification -

$$\text{let } f(n_1, m_1) = f(n_2, m_2)$$

$$\Rightarrow 2^{n_1} 3^{m_1} = 2^{n_2} 3^{m_2}$$

$$\Rightarrow 2^{n_1 - n_2} = 3^{m_2 - m_1}$$

$$\Rightarrow n_1 - n_2 = 0, m_2 - m_1 = 0$$

$$\Rightarrow \boxed{n_1 = n_2} \quad \boxed{m_2 = m_1}$$

$$\therefore (n_1, m_1) = (n_2, m_2)$$

Hence $f(n, m)$ is an injective mapping

$\therefore B = \mathbb{N} \times \mathbb{N}$ is countable infinite set

$\therefore f$ is injective

होता है one-one

$$\text{ए } f(x) = f(y)$$

$$x = y$$

$$\text{अ } \exists x, f(x) \neq f(y)$$

$$\Rightarrow x \neq y$$

$$2^3 \neq 3^3$$

$$2^0 = 3^0$$

Q Prove that $(0,1)$ is uncountable.

proof let $I = (0,1)$

Then we have to show that I is uncountable

If possible let I is countable then we can rewrite the element of I in following order

$$I = \{x_1, x_2, x_3, x_4, \dots\}$$

$$x_1 = 0.x_{11} x_{12} x_{13} \dots$$

$$x_2 = 0.x_{21} x_{22} x_{23} \dots$$

$$x_3 = 0.x_{31} x_{32} x_{33} \dots$$

$$x_4 = 0.x_{41} x_{42} x_{43} \dots$$

$$\text{here } x_{ij} \in \{0, 1, 2, \dots, g\}$$

$$\forall i, j \in \mathbb{N}$$

$$\text{define } x = 0.b_1 b_2 b_3 \dots$$

$$b_i \in \{0, 1, 2, \dots, g\}$$

$$b_1 \neq x_{11}$$

$$b_2 \neq x_{22}$$

$$b_3 \neq x_{33}$$

$$\vdots$$

clearly $x \in (0,1)$
but $x \neq x_n \quad \forall n \in \mathbb{N}$
which is not possible
so ^{our} assumption is wrong
hence $(0,1)$ is uncountable.

* $\mathbb{R}$ is uncountable.

proof - If possible let $\mathbb{R}$ is countable we know that every subset of countable set is countable & $(0,1) \subset \mathbb{R} \Rightarrow (0,1)$ is countable which is false $[\because (0,1)$ is uncountable.]

Hence $\mathbb{R}$ is uncountable.

* $\mathbb{Q}^c$ is uncountable (set of irrationals is uncountable).

proof - We know that $\mathbb{Q}$ is countable if possible let $\mathbb{Q}^c$ is countable.

$\Rightarrow \mathbb{Q} \cup \mathbb{Q}^c$ is countable [union of two countable set is countable.]

$\Rightarrow \mathbb{R}$ is countable.

which is wrong

($\mathbb{R}$ is uncountable)

This $\mathbb{Q}^c$ is countable is not possible.

Hence $\mathbb{Q}^c$ is uncountable.

which is not possible

NOT $\text{If } y \in B \Rightarrow y \in f(y)$
 $\Rightarrow y \in B$

which is again a contradiction

Then our assumption is wrong.

hence any set can not be similar to its power set

* Schroeder Bernstein Theorem:

If A is similar to some subset set B.

& B is similar to some subset of A

then A is similar to B

$$A \sim B_1 \subseteq B$$

$$B \sim C \subseteq A$$

$$B \sim C \subseteq A$$

$$B \sim C \subseteq A$$

$$B \sim C \subseteq A$$

$$|A| < |P(A)|$$

* Any set can not be similar to its power set

Let A be any set & $P(A)$ is power set
Then we have to show that A is
not similar to $P(A)$. If possible let
 $A \sim P(A)$

$\Rightarrow \exists$ a one-one & onto map $f: A \rightarrow P(A)$

Clearly for any $x \in A \Rightarrow f(x) \in P(A)$

$$\Rightarrow f(x) \subseteq A$$

So for any $x \in A$ we have only two
conditions either $x \in f(x)$ or $x \notin f(x)$

Define $B = \{x \in A : x \in f(x)\}$

clearly $B \subseteq A$

$$\Rightarrow B \in P(A)$$

since f is onto

$\Rightarrow \exists y \in A$ such that

$$f(y) = B$$

have $y \in B$ or $y \notin B$

case I if $y \in B \Rightarrow y \in f(y)$

$$\Rightarrow y \notin B$$

which is not possible

Case II

$$\text{If } y \notin B \Rightarrow y \notin f(y) \\ \Rightarrow y \in B$$

which is again a contradiction.

Thus our assumption is wrong.

hence any set can not be similar to its power set.

* Schroeder Bernstein Theorem:

If A is similar to some subset set B .

& B is similar to some subset of A
 then A is similar to B .

$$A \sim B_1 \subset B$$

$$|A| = |B_1| \leq |B| \quad \text{--- (i)}$$

$$B \sim A_1 \subset A$$

$$|B| = |A_1| \leq |A| \quad \text{--- (ii)}$$

$$\text{① \& ② } |A| = |B|$$

* Topology on X .

Let X be any set and T be any subset of $P(X)$. Then T is called Topology on X if:

$$(1) \phi, X \in T$$

$$(2) \text{ if } A, B \in T \Rightarrow A \cap B \in T$$

$$(3) \text{ for any } I \in \Lambda, A_I \in T \Rightarrow \bigcup_{I \in \Lambda} A_I \in T$$

And (X, T) is known as topological space.

* Interior point of topological space:-

Let (X, T) be any topological space and $x \in X$.

x is called interior point of $A \subseteq X$

if $\exists G \in T$ such that

$$x \in G \subseteq A$$

x is called interior point of A

A is called neighbourhood (nbd) of x .

Note: In a topological space any $G \in T$ is called T -open subset of X .

$T \in \{A \in \mathcal{P}(N) \mid A^c \text{ is finite}\}$

Thus show that T is topology on N .

Note - in place of N any set X can be taken.

Q.4 let R be set of real number [semi open topology]
 $T = \{(x, x+\epsilon) : x \in R, \epsilon > 0\} \cup \{\emptyset, R\}$
Then show that T is a topology on R

Blue ① $[T_1] : \emptyset, A \in T$

$$[T_2] : \emptyset \cap \emptyset = \emptyset \in T$$

$$\emptyset \cap A = \emptyset \in T$$

$$A \cap A = A \in T$$

$$[T_3] : \emptyset \cup A \in T$$

Clearly T holds all condition for being a topology.

(2)

$$2) [T_1]: \phi, A \subseteq A \Rightarrow \phi, A \in P(A)$$

$$[T_2]: \text{let } A_1, A_2 \in P(A) \Rightarrow A_1 \subseteq A \text{ \& } A_2 \subseteq A$$

$$\Rightarrow A_1 \cap A_2 \subseteq A \cap A$$

$$\Rightarrow A_1 \cap A_2 \subseteq A$$

$$\Rightarrow A_1 \cap A_2 \in P(A)$$

$$[T_3] \text{ let } A_i \subseteq A \forall i \in I$$

here I is any suitable index set.

$$\bigcup_{i \in I} A_i \subseteq \bigcup_{i \in I} A \cup A_i = A$$

$$\Rightarrow \bigcup_{i \in I} A_i \in P(A)$$

$$(3) N^c = N - N = \phi$$

clearly N^c is finite

per internet. $\Rightarrow N \in T$

is only. Now $\phi^c = N - \phi = N$

here ϕ^c is infinite.

$$\Rightarrow \phi \notin T$$

so T is not a topology on N

Q. let be any set and T be a finite topology on A , Then A is

(a) finite (b) infinite.

(3) cofinite topology X be any set.
 $T = \{A \in P(X) : A^c = X - A \text{ is finite}\} \cup \{\emptyset\}$
 N - cofinite topology.

(1) $\emptyset \in T$
 $N^c = N - N = \emptyset$ (finite)
 $N \in T$

(2) let $A, B \in T$

If any one of A or B is $\emptyset$
 then $A \cap B = \emptyset \in T$

$A \neq \emptyset, B \neq \emptyset$

$\Rightarrow A^c, B^c$ are finite

$\Rightarrow A^c \cup B^c$ is finite

$\Rightarrow (A \cap B)^c$ is finite

$\Rightarrow A \cap B \in T$

(3) $A_i \in T \forall i \in \mathbb{N}$

$\Rightarrow A_i^c$ is finite $\forall i \in \mathbb{N}$

$\Rightarrow \bigcap_{i \in \mathbb{N}} A_i^c$ is finite

$\Rightarrow \left(\bigcup_{i \in \mathbb{N}} A_i \right)^c$ is finite $\Rightarrow \bigcup_{i \in \mathbb{N}} A_i \in T$

Q.5 let $X = \{a, b, c, d, e\}$ which of the following topology on X

(i) $T_1 = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}$

(ii) $T_2 = \{X, \emptyset, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}$

(iii) $T_3 = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c, d\}, \{a, b, c, d\}\}$

$X = \{1, 2\}$
 How many distinct topologies on X
 can exist.

$$[\tau_1] = \{\phi, X\}$$

$$[\tau_2] = P(X)$$

Types of topology -

(i) Indiscrete topology - if $\tau = \{\phi, X\}$ the
 (X, τ) is said to be
 indiscrete topology

ex. $X = \{1, 2, 3\}$

$$\tau = \{\phi, \{1, 2, 3\}\}$$

$\{X, \tau\}$ is called indiscrete topology.

ex. (i) $\phi, X \in \tau$

(ii)

τ	ϕ	$\{1, 2, 3\}$
ϕ		
$\{1, 2, 3\}$		

(iii)

τ	ϕ	$\{1, 2, 3\}$
ϕ		
$\{1, 2, 3\}$		

(ii) Discrete topology - If τ is a collection of all
 subsets of X , then (X, τ) is
 said to be discrete topology.

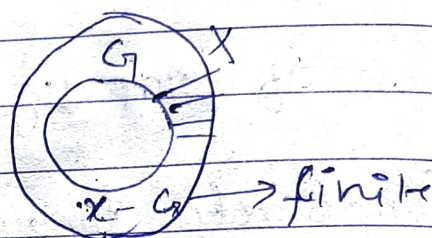
ex. $X = \{1, 2, 3\}$

$$\tau = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$$

then (X, T) is said to be discrete topology.

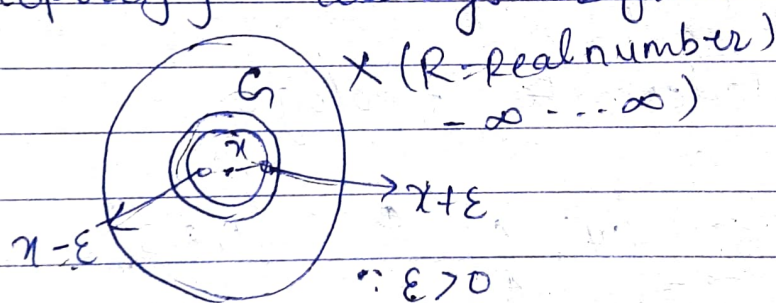
(iii) Co-finite topology - If $T = \{G \mid G \subseteq X \text{ and } X - G \text{ is finite}\}$ then (X, T) is said to be co-finite topology space.

that $X - G$ is complement of G .



(iv) Usual topology - If $T = \{G \subseteq X \mid \text{for each } x \in G, (x - \epsilon, x + \epsilon) \subseteq G\}$

then T is called usual topology on X . usual topology always defined in $\mathbb{R}$.



(v) Weaker & Strong topology - let T_1 and T_2 be two topologies

on X , if $T_1 \subseteq T_2$

then T_1 is called weaker than T_2

and T_2 is called stronger than T_1 .

e.g. $X = \{1, 2, 3\}$

$T_1 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 2, 3\}\}$

$$T_2 = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\} \}$$

$$\therefore T_1 \subset T_2$$

Q. proved - that intersection of topologies is a topologies but not necessary that union of two topologies is a topology.

$$T_1 \cup T_2 \neq T$$

$$T_1 \cap T_2 = T$$

proof:

let T_1 and T_2 are two topologies on X
 $\therefore T_1$ is a topology.

$$(i) \quad \emptyset, X \in T_1$$

(ii) T_1 is closed under arbitrary union

$$\text{if } A, B \in T_1 \Rightarrow A \cup B \in T_1$$

$$\text{for only } \lambda \in \Lambda, A_\lambda \in T_1 \Rightarrow \bigcup_{\lambda \in \Lambda} A_\lambda \in T_1$$

(iii) T_1 is closed under finite intersection

$$\text{if } A, B \in T_1 \Rightarrow A \cap B \in T_1$$

and $\therefore T_2$ is a topology.

$$(i) \quad \emptyset, X \in T_2$$

(ii) T_2 is closed under arbitrary union.
 for any $I \in \Lambda$, $A_i \in T_2 \Rightarrow \bigcup_{i \in \Lambda} A_i \in T_2$

(iii) T_2 is closed under finite intersection.
 if $A, B \in T_2 \Rightarrow A \cap B \in T_2$

to prove that $T_1 \cap T_2$ is a topology on X .

(i) $\phi, X \in T_1 \cap T_2$
 $\because \phi, X \in T_1$ and $\phi, X \in T_2$
 $\therefore \phi, X \in T_1 \cap T_2$

(ii) $T_1 \cap T_2$ is closed under arbitrary union.
 let $G_i \in T_1 \cap T_2 \quad \forall i \in \Lambda$
 $\therefore G_i \in T_1$ and $G_i \in T_2$
 $\therefore \bigcup_{i \in \Lambda} G_i \in T_1$ and $\bigcup_{i \in \Lambda} G_i \in T_2$
 $\therefore \bigcup_{i \in \Lambda} G_i \in T_1 \cap T_2$

(iii) $T_1 \cap T_2$ is closed under infinite intersection,
 let $A, B \in T_1 \cap T_2$
 $\Rightarrow A \in T_1$ and $A \in T_2$
 $\& B \in T_1$ and $B \in T_2$

$$\Rightarrow A \cap B \in T_1 \text{ and } A \cap B \in T_2$$

$$\Rightarrow A \cap B \in T_1 \cap T_2$$

Hence $T_1 \cap T_2$ is closed under finite intersection

Hence $T_1 \cap T_2$ is a topology, but union of two topologies is not a topology on X .

$$\text{ex - } X = \{1, 2, 3\}$$

$$T_1 = \{ \emptyset, \{1\}, \{2\}, \{1, 2, 3\}, \{1, 2\} \}$$

$$T_2 = \{ \emptyset, \{1\}, \{3\}, \{1, 3\}, \{1, 2, 3\} \}$$

$$\text{and } T_1 \cup T_2 = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\} \}$$

Hence T_1 and T_2 are two topologies

but $T_1 \cup T_2$ is not a topology.

Hence proved.

$$\Rightarrow A \cap B \in T_1 \text{ and } A \cap B \in T_2$$

$$\Rightarrow A \cap B \in T_1 \cap T_2$$

Hence $T_1 \cap T_2$ is closed under finite intersection.

Hence $T_1 \cap T_2$ is a topology, but union of two topologies is not a topology on X .

$$\text{ex - } X = \{1, 2, 3\}$$

$$T_1 = \{ \emptyset, \{1\}, \{2\}, \{1, 2, 3\}, \{1, 2\} \}$$

$$T_2 = \{ \emptyset, \{1\}, \{3\}, \{1, 3\}, \{1, 2, 3\} \}$$

$$\text{and } T_1 \cup T_2 = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\} \}$$

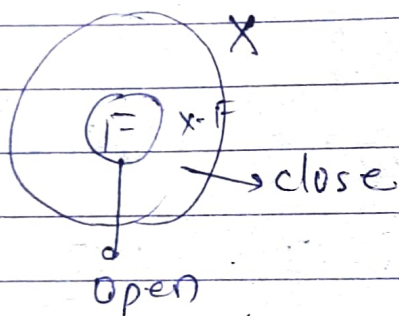
Hence T_1 and T_2 are two topologies

but $T_1 \cup T_2$ is not a topology.

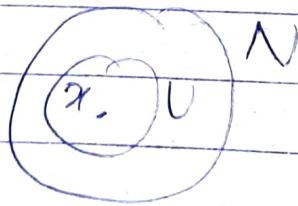
Hence proved.

* Open set - If (X, τ) be a topological space, then every member of τ is open set in X .

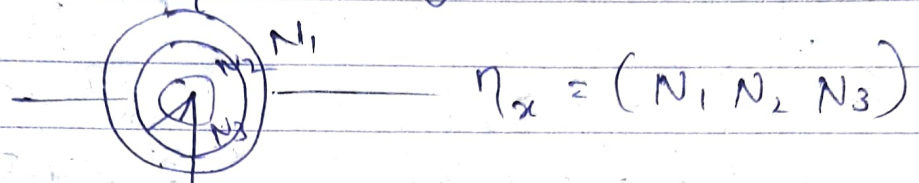
* Close set - A set F is said to be closed set, if its complement $X - F$ is open.



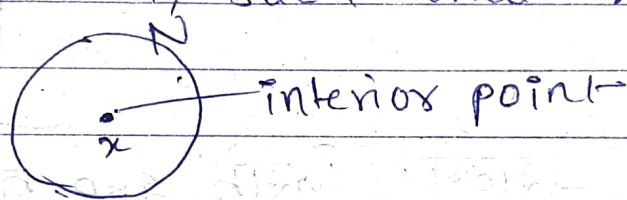
Neighbourhood - (nbd) - Let (X, τ) be a topological space and $x \in X$. Then, a subset N of X is said to be nbd of x if there exists some open set U in X such that $x \in U \subseteq N$.



- Neighbourhood system - collection of all neighbourhood. we show that η_x where $\eta_x = \text{neighbourhood system}$.



- Interior point - A point x is said to be interior point of N , if $\exists$ an open set G , such that $x \in G \subset N$



G° - set of interior points

A collection of all interior point of G .

Note :- A° - all interior points of A .
 G° is a open set.

- closure of a set -

$$(\bar{A}) = (A \cup A')$$

$$= (A \cup \text{limit point of } A)$$

$$= [A \cup (\text{Accumulation point of } A)]$$

$$= [A \cup (\text{clupture point of } A)]$$

$$= [A \cup (\text{contensation point of } A)]$$

limit $\rightarrow$

function sequence
in time use

limit point $\rightarrow$
set of limit use

(ii) The intersection of all closed superset is called a closure of set A .

$$A = \{1, 2, 3\}$$

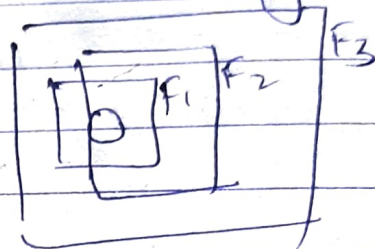
$$B = \{1, 2, 3, 4\}$$

$$C = \{1, 2, 3, 4, 5\}$$

$$D = \{1, 2, 3, 4, 5, 6\}$$

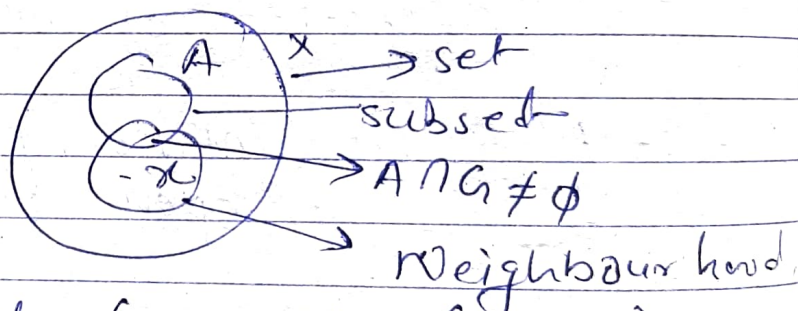
F_1, F_2, F_3 are A or superset of A

$\bar{A} = \cap F$, where $F \supseteq A$ and F being closed.



• Dense set - (सघन) जिसमें gap न हो।

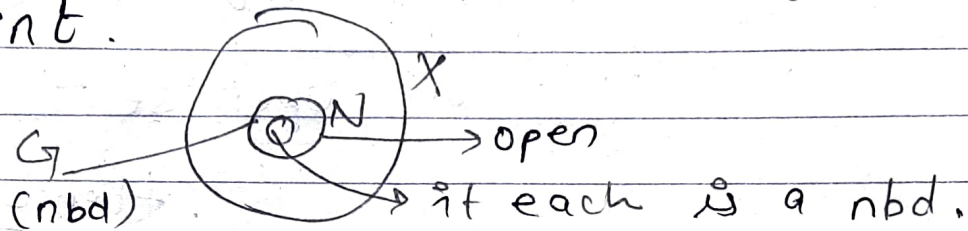
(i) A set A is said to be dense set in X if $A \cap G \neq \emptyset$



for every (x) open set G of X , $(G \cap X)$

(ii) A set A is said to be dense set in X if closure of A $(\bar{A}) = X$

Q. → proof that a subset of a topological space is open iff it is a nbd of each its point.



proof - $\because$ let (X, T) be a topological space.
and $N \subset X$

suppose that N is open set.

$$\Rightarrow x \in G \subset N$$

i.e. G is a nbd of x but

$x \in G$ is arbitrary

Hence G is a nbd of each of its point conversely.

suppose that G is a nbd of each of its points.

let $x \in G$ be arbitrary.

then $x \in G$ is an nbd in N .

By def. of nbd :- $\exists$ an open set N such that $x \in G \subset N$

$$\Rightarrow \{x\} \subset G \subset N$$

$$\Rightarrow \cup \{x\} \subset \cup G \subset \cup N$$

$$\Rightarrow N \subset \cup G \subset N$$

$$\Rightarrow N = \bigcup G$$

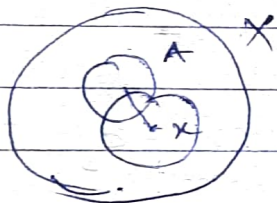
$N = \text{Union of open set}$

$N = \text{an open set}$

Hence proved

limit point :-

Let A be a finite subset of X , then a point $x \in X$ is said to be limit point of A , if neighbourhood of x contain a point of A .



$X = \text{set}$

$A = \text{subset}$

x एक ऐसा point है - जिसके किसी भी radius के circle को भी जो A subset के एक भी point को ले लेता है वह x , limit point कहलाता है।

Theorem:- let (X, T) is a topological space and let A, B be subset of X . then prove that.

$$(i) \quad \overline{\phi} = \phi$$

$$(iv) \quad A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$$

$$(ii) \quad \overline{A \cup B} = \overline{A} \cup \overline{B}$$

$$(vi) \quad (\overline{\overline{A}}) = A$$

$$(iii) \quad A = \overline{\overline{A}}$$

$$(iv) \quad \overline{A \cap B} = \overline{A} \cap \overline{B}$$

proof:- ① $\overline{\phi} = \phi$

$\therefore \phi$ is a closed set.

$\therefore \phi$ is a closed set.

then $\overline{\phi} = \phi$

ϕ सदैव closed

और open दोनों

हो सकता है।

② $\therefore A \subseteq \overline{A}$ and $B \subseteq \overline{B} \rightarrow \overline{A}$ एक ऐसा smallest set है जो A को खता है

$$A \cup B \subseteq \overline{A} \cup \overline{B} \text{ --- (1)}$$

and $A \cup B \subseteq \overline{A \cup B} \text{ --- (2)}$ $\overline{A \cup B} \supseteq \overline{A} \cup \overline{B}$ की खता है

Now from ① $\overline{A \cup B}$ is the smallest closed set containing $A \cup B$.

But from eq. ② $\overline{A \cup B}$ is the smallest closed containing $A \cup B$ from eq ① & ②

$$\therefore \overline{A \cup B} \subseteq \overline{A} \cup \overline{B} \text{ --- (3)}$$

$\therefore A \subseteq A \cup B$ and $B \subseteq A \cup B$

$\therefore \overline{A} \subseteq \overline{A \cup B}$ and $\overline{B} \subseteq \overline{A \cup B}$

$$\bar{A} \cup \bar{B} \subset \overline{A \cap B} \cup \overline{A \cap B}$$

$$\bar{A} \cup \bar{B} \subset \overline{A \cap B} \quad \text{--- (4)}$$

$$\therefore \boxed{\overline{A \cap B} = \bar{A} \cup \bar{B}}$$

$$\begin{cases} x < y, y < x \\ \Leftrightarrow x = y \end{cases}$$

$$(11) \quad \bar{\bar{A}} = A$$

$\therefore \bar{A}$ is the smallest closed set containing

$$\therefore A \subset \bar{A} \quad \text{--- (1)}$$

$$\text{let } A \subset B \quad \text{--- (2) and } B \subset \bar{B}$$

$$\therefore A \subset \bar{B} \quad \text{--- (3)}$$

thus $\bar{B}$ is the smallest set containing A

$$\left. \begin{array}{l} \text{But } A \subset B \\ \bar{A} \subset \bar{B} \quad \text{--- (4)} \\ \text{from eqn (3) \& (4)} \end{array} \right\}$$

But $\bar{A}$ is the smallest closed set, Hence

$$\boxed{\bar{A} = A}$$

$$(10) \quad \overline{A \cap B} \subset \bar{A} \cap \bar{B}$$

$$\because A \cap B \subset A \text{ and } A \cap B \subset B$$

$$\therefore \overline{A \cap B} \subset \bar{A} \text{ and } \overline{A \cap B} \subset \bar{B}$$

$$\therefore (\overline{A \cap B}) \cap (\overline{A \cap B}) \subset \bar{A} \cap \bar{B}$$

$$\overline{A \cap B} \subset \bar{A} \cap \bar{B}$$

(V) $A \subset B \Rightarrow \bar{A} \subset \bar{B}$

$\because A \subset \bar{A}$ and $B \subset \bar{B}$ — (1)

Let $A \subset B$; $B \subset \bar{B}$

$\therefore A \subset \bar{B}$ — (2)

But $\bar{B}$ is closed set.

thus $\bar{B}$ is the closed set containing A — (3)

$\therefore \bar{A}$ is the closed set containing A — (4)

$\bar{A}$ is the smallest-closed set

$$A \subset \bar{A} \subset \bar{B}$$

$\therefore \bar{A} \subset \bar{B}$ Hence proved.

(VI) $(\bar{\bar{A}}) = \bar{A}$

By

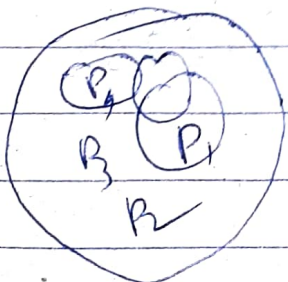
result " $\bar{A}$ is the smallest closed set containing A ".

we have proof that $\Rightarrow \bar{\bar{A}} = \bar{A}$

$$\boxed{(\bar{\bar{A}}) = \bar{A}}$$

Hence proved.

Derived set :- Derived set is the collection of all limit points of A and it is denoted by $D(A)$.



$$P = P_1 + P_2 + P_3 + P_4 \dots$$

* Base for topology :- let (X, T) is a topological space, and $\beta \in T$ then β is said to be base for T , "If every member of T can be express by the union of sum member of β "

ex. $X = \{1, 2, 3\}$

and $T = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

$\beta = \{\emptyset, \{1\}, \{2\}, \{3\}\}$

Subbase for topology - let (X, T) be a topological space and β be a base for T , and let S is a subset of T , S is said to be subbase for T , if every member of β can be express as the finite intersecting of member of S . that is

$B = \beta$

then $B = \bigcap_{i=1}^n S_i$

ex. let $X = \{1, 2, 3\}$

and $T = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

$\beta = \{\emptyset, \{1\}, \{2\}, \{3\}\}$

$$S = \{ \emptyset, \{1,2\}, \{1,3\}, \{2,3\} \}$$

Note - $T = \{G \mid G = \cup B \text{ for some } B \in \beta\}$
 and $\beta = \{B \mid B = \bigcap_{i=1}^n S_i, S_i \in S\}$

Note - S generates T means T is smallest topology containing S .

(18)
 theorem:- let (X, T) be a topological space and S is a family of subsets of X then S is a subbase for T iff S generates T .

proof - Suppose S is a subbase for T .
 to prove that - S generates T .

$$\therefore T = \{G \mid G = \cup B \text{ and } B \in \beta\}$$

$$\text{and } \beta = \{B \mid B = \bigcap S_i \text{ and } S_i \in S\}$$

$$\therefore S \subset \beta \text{ and } \beta \subset T$$

$$\therefore S \subset T$$

let u be another topology containing S
 i.e. $S \subset u$

$\therefore u$ is a topology.

then $\mathcal{u}$ is closed under finite intersection.

$$\beta \subset \mathcal{u}$$

$\therefore \mathcal{u}$ is a topology.

$\therefore \mathcal{u}$ is closed under arbitrary union.

$$\mathcal{T} \subset \mathcal{u}$$

Hence $\mathcal{T}$ is smallest topology containing $\mathcal{S}$.
 $\therefore \mathcal{S}$ generates $\mathcal{T}$.

Conversely :- suppose that $\mathcal{S}$ generates $\mathcal{T}$.
 $\mathcal{T}$ is the smallest topology containing $\mathcal{S}$.

to prove that $\mathcal{S}$ is a subbase for $\mathcal{T}$.
 let $\mathcal{u}$ be another topology such that

$\mathcal{S}$ is a subbase for $\mathcal{u}$.

then $\mathcal{u} = \{G \mid G = \bigcup B \text{ for some } B \in \beta\}$

$$\text{and } \beta = \{B \mid B = \bigcap S_i, S_i \in \mathcal{S}\}$$

$\therefore \mathcal{T}$ is a topology.

$\therefore \mathcal{T}$ is closed under finite intersection.

$$\beta \subset \mathcal{T}$$

and T is closed under arbitrary union.

$$u \subset T \quad \text{--- ①}$$

but T is smallest topology containing S .

$$\text{i.e. } T \subseteq u \quad \text{--- ②}$$

$$u = T$$

$\therefore S$ is a subbase for T .

Relative topology / subspace - ① let (X, T) be topological space

and $Y \subset X$, then (Y, T_Y) is a relative topology or subspace on X .

ex. $X = \{1, 2, 3\}$

$$T = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\} \}$$

let $Y = \{1, 2\}$

$$T_Y = \{ \emptyset, \{1\}, \{2\}, \{1, 2\} \}$$

$$\text{then } (Y, T_Y) \subset (X, T)$$

② let (X, T) be a topological space and $Y \subset X$.
Then $T_Y = \{ Y \cap G \mid G \in T \}$

is called relative topology on y and (y, T_y)
 is called subspace of (X, T)

ex. $y = \{1, 2\}$

$$T_y = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\} \}$$

$$T_y = \{ y \cap G \mid G \in T \}$$

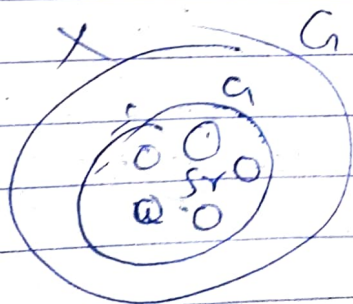
$$T_y = \{ \emptyset, \{1\}, \{2\}, \{1, 2\} \}$$

③ Let (X, T) be a topological space and y is a subset of X . the relative topology T_y is the family those subsets of y , then

$$T_y = \{ G \cap y : G \in T \}$$

then topological space (y, T_y) is called subspace of (X, T)

Open set - A set G is said to be open set in X , if



$$G = \bigcup S_x(x), \quad x \in X$$

where $S_x(x) = \text{Neighbourhood}$
 $= N_r(x)$

② A set G is said to be open set in X , if $\forall x \in X$, such that $x \in S_r(x) \subset G$

③ A set G is said to be open set in X , if it is a nbd of each its points.

④ A set G is said to be open set in X , if $X - G$ is closed.

Theorem:- let A a subset of a topological space, then prove that $\bar{A} = A \cup A'$

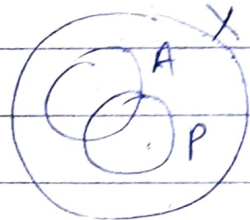
OR

where $D(A)$ = derived set. $\bar{A} = A \cup D(A)$

proof - let (X, T) be a topological space and A is a subset of X .

let p be the limit point of A .

then $p \in A'$ — (1)



$D(A)$

$\therefore \bar{A}$ is the smallest closed set containing A i.e. $A \subset \bar{A}$ — (2)

By the result,

" a subset of A of a topological space X is closed iff A contains all its limit points."

thus $\bar{A}$ contains all its limit points.
 $\Rightarrow A' \subset \bar{A} \quad \text{--- (3)}$

$$\therefore A \cap \bar{A} \text{ and } A' \cap \bar{A}$$

$$A \cup A' \subset A \cup \bar{A}$$

$$A \cup A' \subset \bar{A} \quad \text{--- (4)}$$

we now suppose that

$$\bar{A} \subset A \cup A'$$

if possible $\bar{A} \not\subset A \cup A'$

let $p \in \bar{A}$ but $p \notin A \cup A'$

$p \in \bar{A}$ and $p \notin A$ or $p \notin A'$

which is a contradiction because p is limit point of A .

$$\therefore \bar{A} \subset A \cup A' \quad \text{--- (5)}$$

from eqⁿ (4) & (5)

$$\boxed{\therefore \bar{A} = A \cup A'} \quad \text{Hence proved}$$

★ Zorn's lemma :- Every non-void partial ordered set in which chain has an upper bound has a maximal element.

proof :- suppose $(P, \leq)$ is a non-void partial ordered set.

first we show that P contains maximal chain.

let $\mathcal{C}$ be the family of all chain in P . then $\mathcal{C}$ is a non-void family of finite character as follows.

If $A \in \mathcal{C}$ and B be any finite subset of A then $x, y \in B \Rightarrow x, y \in A$ [$\because B \subseteq A$]

$\Rightarrow$ either $x \leq y$ or $y \leq x$

$\Rightarrow B$ is a chain.

$\Rightarrow B \in \mathcal{C}$

thus every finite subset of A is in $\mathcal{C}$.
 Conversely,

suppose that every finite subset of A is in $\mathcal{C}$.

then $x, y \in A \Rightarrow \{x, y\}$ is a finite subset of A .

$\Rightarrow \{x, y\} \in \mathcal{C}$

- $\Rightarrow$ either $x \leq y$ or $y \leq x$
- $\Rightarrow A$ is a chain
- $\Rightarrow A \in C$

Thus if every finite subset of A is in C then A is also in C .

Hence C is finite character

By ~~tukey~~ lemma,

"Any family C of finite character has a maximal element".

Thus C has a maximal element.

Hence P contains a maximal chain.

Again Assume that $(P, \leq)$ is a non-void partial ordered set in which each chain has an upper bound, &

say u .

then $\exists$ a maximal chain M in P

$\therefore u$ is an upper bound of M .

$\therefore u$ is the maximal element of P .

this complete proof.

* (23)

Well ordering theorem:- Every set can be well ordered i.e. for every set S , $\exists$ some well ordering $\leq$ on S .

or

A non-empty finite ordered set has the order type of a section $\{1, 2, 3, \dots, r\}$ of $\mathbb{Z}^+$, so if S is non-empty well ordered.

proof - let S be any non-void set. we wish to find a well-ordering $\leq$ on S .

let $\mathcal{S}$ denote the family of all well ordered $(W, \leq)$ such that $W \subseteq S$

we denote an ordering $\leq$ on W as follows

$(W_1, \leq_1) \leq (W_2, \leq_2)$ iff either

$$W_1 = W_2 \quad \text{or} \quad \leq_1 \leq \leq_2$$

$$\leq_1 \neq \leq_2$$

it can be easily seen that $\leq_1$ is a partially ordering in W .

We will show that partial ordered set $(W, \leq)$ satisfies the Zorn's lemma.

Let $\mathcal{C} = \{ (w_i, \leq_i) : i \in I \}$ be any non-void chain in $\mathcal{W}$.

Write $w = \bigcup_{i \in I} w_i$ and $\leq = \bigcup \{ \leq_i : i \in I \}$

then clearly.

$(w, \leq)$ is an upper bound of $\mathcal{C}$ thus by zorn's lemma.

the set $(w, \leq)$ has a maximal element

let $(w_0, \leq_0)$ be the maximal element of $(w, \leq)$

Case (i) If $w_0 = S$

then $\leq_0$ is well ordering on S and so S is well ordered.

Case (ii) If $w_0 \neq S$

Then $z \in G$ But $z \notin w_0$

taking $w_1 = w_0 \cup \{z\}$

and define a well ordering $\leq'$ on w' .

then $(w', \leq')$ is a well ordered set.

i.e. $(W', \leq') \in W$

But $(W_0, \leq_0) \leq (W', \leq')$

this contradicts

the maximality of $(W_0, \leq_0)$

thus $W_0 = S$

hence $\leq_0$ is a well ordering on S
and so S is a well ordering.

Cantor's theorem: ⁽²²⁾

statement -

No set X can be equivalent to its power set $P(X)$.

proof:- let X be a non-empty set.

and $P(X)$ be a power set of X .

i.e. $P(X)$ is a collection of all subset of X .

let A is a subset of $P(X)$,

and it is equivalent to X .

then we have to show that

$$\boxed{A \neq P(X)}$$

let $f: X \rightarrow A$ be one-one and onto mapping
so that $X \sim A$

we define $S = \{x \in X \mid x \notin f(x)\}$

$\therefore f$ is onto.

$\therefore$ for every $A \in \mathcal{A} \exists x \in X$ such that

$$f(x) = A$$

(i) If $x \notin f(x)$ then $x \in S$ and $x \in A$
 $S \neq A$

thus we have shown that
there is a subset S of X such
that $S \neq A$.

$$\text{this} \Rightarrow S \notin \mathcal{A}$$

Here $S \in \mathcal{P}(X)$ But $S \notin \mathcal{A}$

$$\therefore A \neq \mathcal{P}(X)$$

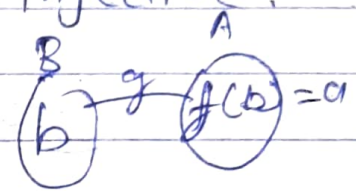
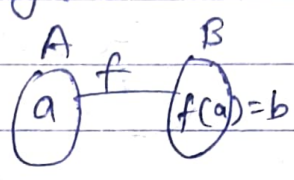
Here no set X can be equivalent
to its power set $\mathcal{P}(X)$.

(19)

★ Schroeder Bernstein theorem:-

let A and B be two sets. let $f: A \rightarrow B$ and $g: B \rightarrow A$ is injective mapping, then $\exists$ a bijective $\psi: A \rightarrow B$

proof - let A and B are two sets and $f: A \rightarrow B$, $g: B \rightarrow A$ is injective mapping

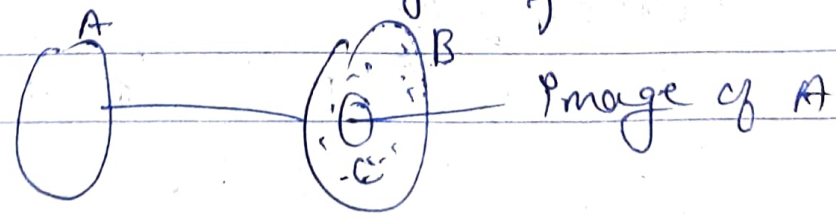


case(i) if f is surjective mapping and f is also injective.
 $\therefore f$ is a bijective mapping.

case(ii) if g is surjective mapping and g is also injective.
 $\therefore g$ is a bijective mapping.

$\therefore g^{-1}: A \rightarrow B$ is also a surjective mapping

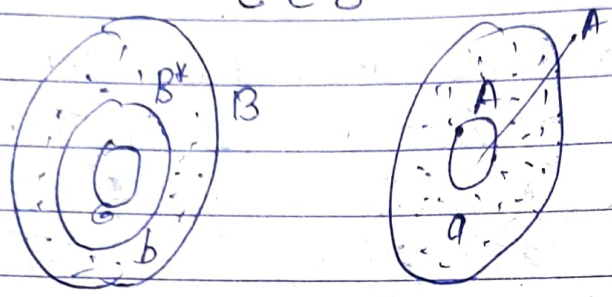
case(iii) if assume that neither f is surjective nor g is surjective.
 let $C = B - \text{image of } A$



write $B^* = \{ (fg)^n(c) \mid c \in C, n \geq 0 \}$
 and

$$A^* = \{ g(B^*) \} = \{ g(b) \mid b \in B^* \}$$

and $c \in B^*$



$$\therefore g: B^* \rightarrow A^*$$

$$g: B^* \rightarrow g(B^*)$$

$\therefore g$ is a surjective mapping and g is injective mapping

i.e. $g: B^* \rightarrow A^*$ is a bijective mapping

then $g^{-1}(a) \in B^*$

we define a mapping

$$\psi: A \rightarrow B$$

$$\psi(a) = \begin{cases} g^{-1}(a) & \text{if } a \in A^* \\ f(a) & \text{if } a \notin A^* \end{cases}$$

claim :- ψ is bijective mapping.

verification :- we first show that $a \in A^*$
iff $f(a) \in B^*$

$$a \in A^* \Leftrightarrow a = g(b) \text{ for some } b \in B^*$$

$$\therefore g f(a) = f(g(b)) \quad [\because A^* = g \in B^*]$$

$$= fg(b) \quad \text{and} \quad b \in (fg)^n(c)$$

$$= fg(fg)^n(c)$$

$$f(a) = (fg)^{n+1}(c),$$

$$f(a) \in B^*$$

where $c \in C$

(i) ψ is an injective mapping

let a and $a' \in A$

$$\text{let } \psi(a) = \psi(a')$$

$$\Rightarrow a = a'$$

(a) if a and $a' \in A^*$

$$\psi(a) = g^{-1}(a)$$

$$\psi(a') = g^{-1}(a')$$

$$\therefore \psi(a) = \psi(a')$$

$$g^{-1}(a) = g^{-1}(a')$$

$$a = gg^{-1}(a')$$

$$\boxed{a = a'}$$

claim :- ψ is bijective mapping.

verification :- we first show that $a \in A^*$
iff $f(a) \in B^*$

$$a \in A^* \Leftrightarrow a = g(b) \text{ for some } b \in B^*$$

$$\therefore g f(a) = f(g(b)) \quad [\because A^* = g \in B^*]$$

$$= fg(b) \quad \text{and} \quad b \in (fg)^n(c)$$

$$= fg(fg)^n(c)$$

$$f(a) = (fg)^{n+1}(c),$$

$$f(a) \in B^*$$

where $c \in C$

(i) ψ is an injective mapping

let a and $a' \in A$

$$\text{let } \psi(a) = \psi(a')$$

$$\Rightarrow a = a'$$

(a) if a and $a' \in A^*$

$$\psi(a) = g^{-1}(a)$$

$$\psi(a') = g^{-1}(a')$$

$$\therefore \psi(a) = \psi(a')$$

$$g^{-1}(a) = g^{-1}(a')$$

$$a = gg^{-1}(a')$$

$$\boxed{a = a'}$$

(b) if $a, a' \in A^*$

$$\therefore \psi(a) = f(a)$$

$$\psi(a') = f(a')$$

$$\therefore \psi(a) = \psi(a')$$

$$f(a) = f(a')$$

$$a = f^{-1}f(a')$$

$$\boxed{a = a'}$$

$\therefore \psi(a)$ is an injective mapping.

(2) ψ is a surjective mapping

let $b \in B^*$

(a) if $b \in B^*$, then

$$a = g(b) \in A^*$$

$$a \in A^*$$

$$\therefore \psi(a) = g^{-1}(a)$$

$$[\because a = g(b)]$$

$$\psi(a) = b$$

(b) if $b \notin B^*$, then $b \notin C$ $[\because C \subset B^*]$

$\Rightarrow b \notin \{B\text{-image of } A\}$

$\Rightarrow b \in \text{image of } A$

$[\because C = B\text{-image of } A]$

$\Rightarrow b \in f(A)$

$f(a) \in C \subset B^*$

$$\Rightarrow f(a) \notin B^*$$

$$[\because b \notin B^*]$$

$$\Rightarrow a \notin A^*$$

$$\psi(a) = f(a)$$

$$\psi(a) = b$$

Hence ψ is surjective
and hence ψ is bijective.

Hence proved.

Q. Show that a subspace of a topological space is itself a topological space.

Let (X, τ_X) be a topological space and (Y, τ_Y) be a subspace of (X, τ_X) , then we have to show that (Y, τ_Y) is itself a topological space (i.e. it satisfies the conditions of topological space).

(1) Let $\{U_\alpha : \alpha \in I\}$ be a sub-collection of τ_Y .

Then corresponding to each U_α , there exists some $V_\alpha \in \tau_X$ such that

$$U_\alpha = V_\alpha \cap Y$$

$$\Rightarrow \bigcup_{\alpha \in I} U_\alpha = \bigcup_{\alpha \in I} (V_\alpha \cap Y)$$

$$= \left(\bigcup_{\alpha \in I} V_\alpha \right) \cap Y$$

Since each $V_\alpha \in T$ and T is a topology, $\bigcup_{\alpha \in I} V_\alpha \in T$. Therefore, by definition of relative topology.

$$\left(\bigcup_{\alpha \in I} V_\alpha \right) \cap Y \in T_Y$$

$$\Rightarrow \bigcup_{\alpha \in I} U_\alpha \in T_Y$$

Closure - let (X, T) be a topological space and let $A \subset X$. then the intersection of all closed set which are superset of A is called the closure of A and is denoted by $\bar{A}$.

(11) let $\{U_i : 1 \leq i \leq n\}$ be a finite sub-collection of T_Y then corresponding to each U_i , there exist some $V_i \in T_X$ such that

$$U_i = V_i \cap Y$$

$$\Rightarrow \bigcap_{i=1}^n U_i = \bigcap_{i=1}^n (V_i \cap Y) = \left(\bigcap_{i=1}^n V_i \right) \cap Y$$

since each $V_i \in T_X$ and T_X is Topology,

$$\bigcap_{i=1}^n V_i \in T_X$$

Therefore, by definition of relative topology,

$$\left(\bigcap_{i=1}^n V_i \right) \cap Y \in T_Y$$

$$\Rightarrow \bigcap_{i=1}^n U_i \in T_Y$$

(ii) Now, $\phi \cap Y = \phi$ and $X \cap Y = Y$

$$\Rightarrow \phi, Y \in T_Y$$

This shows T_Y is itself a topology on Y , hence the subspace Y is a topological space.

★ Kuratowski closure Axioms: - Let (X, T) be a topological space. A closure operation on X is a function $k: P(X) \rightarrow P(X)$ satisfying the following four conditions known as Kuratowski closure axioms -

$$[K_1] : \phi^k = \phi$$

$$[K_2] : A \subset A^k$$

$$[K_3] : (A \cup B)^k = A^k \cup B^k$$

$$[K_4] : (A^k)^k = A^k$$

where A & B are any subset of X .

Theorem - (Kuratowski), let X be any set and let c be Kuratowski closure operator on X , that is a function -

$$c: P(X) \rightarrow P(X) \text{ s.t.}$$

$$[K_1] \quad c(\phi) = \phi$$

$$[K_2] \quad A \subset c(A) \quad \forall A \subset X$$

$$[K_3] \quad c(A \cup B) = c(A) \cup c(B) \quad \forall A \subset X$$

$$[K_4] \quad c(c(A)) = c(A) \quad \forall A \subset X$$

Then $\exists$ a ^{one & only one} unique topology T on X such that for each $A \subset X$, $c(A)$ will be the T -closure of the subset A of X .

proof:- let c be the kuratowski closure operator on X i.e.

$C : P(X) \rightarrow P(X)$ such that,

$$[K_1] \quad \phi^c = \phi$$

$$[K_2] \quad A \subset A^c \quad \forall A \subset X$$

$$[K_3] \quad (A \cup B)^c = A^c \cup B^c \quad \forall B, A \subset X$$

$$[K_4] \quad (A^c)^c = A^c \quad \forall A \subset X$$

let f be the family of subset A of X such that $A^c = A$,

$$\text{i.e. } f = \{A \in P(X) : A \subset X \text{ \& } A^c = A\}$$

$$[F_1][K_1] \text{ let } \phi \in P(X) \text{ and } \phi^c = \phi \\ \Rightarrow \phi \in f$$

$$\text{Again } X \in P(X) \Rightarrow X^c \subseteq X$$

$$[K_2] \quad X \subseteq X^c$$

$$\Rightarrow X = X^c$$

$$\Rightarrow X \in f$$

$$[F_2] \text{ let } A, B \in P(X) \text{ then}$$

$$A^c = A \text{ \& } B^c = B$$

$$A, B \in f \Rightarrow A^c \cup B^c = A \cup B$$

$$\Rightarrow (A \cup B)^c = A \cap B$$

$$\Rightarrow A \cup B \in F$$

{ by [K₃] }

[F₃] let $A_i \in F \quad \forall i \in I$

Then $A_i^c = A_i \quad \forall i \in I$ — (1)

we have to prove that, $\cap \{A_i : i \in I\} \in F$
 by [K₂].

$$\cap \{A_i : i \in I\} \subset \left(\cap \{A_i : i \in I\} \right)^c \text{ — (2)}$$

we have to show that,

$$c \left(\cap \{A_i : i \in I\} \right) \subset \cap \{A_i : i \in I\}$$

we have,

$$A \subset B \Rightarrow (A)^c \subset (B)^c \text{ — (3)}$$

Now, $A \subset B \Rightarrow B = A \cup B$

$$\Rightarrow B^c = (A \cup B)^c$$

$$\Rightarrow B^c = A^c \cup B^c \quad [K_3]$$

$$\Rightarrow A^c \subset B^c$$

Now since

$$\cap \{A_i : i \in I\} \subset A_i \quad \forall i \in I \text{ we have } \textcircled{3}$$

$$\left(\bigcap \{A_i : i \in I\} \right)^c \subset A_i^c$$

$$\left(\bigcap \{A_i : i \in I\} \right)^c \subset A_i \quad \text{by (1)}$$

$$\left(\bigcap \{A_i : i \in I\} \right)^c \subset \bigcap \{A_i : i \in I\} \quad \text{--- (4)}$$

from eqn (2) & (4)

$$\left(\bigcap \{A_i : i \in I\} \right)^c = \bigcap \{A_i : i \in I\}$$

$$\bigcap \{A_i : i \in I\} \in F$$

Hence, $A_i \in F \quad \forall i \in I \Rightarrow \bigcap \{A_i : i \in I\} \in F$

Now let T be family consisting of complements of members of F .

$$\text{i.e. } T = \{B : B^c \in F\}$$

$\Rightarrow T$ is unique topology on X .

Now we have to prove, $A^c = \bar{A}$

$$[K_4], \quad (A^c)^c = A^c \Rightarrow A^c \in F$$

$$[K_2], \quad A \subset A^k \quad \text{--- (5)}$$

Let $A \subset B$ then $B \in F \Rightarrow B^c = \bar{B}$

Also, $A \subset B \Rightarrow A^c \subset B^c$ —

then $A^c \subset B$ — (6) $[B^c = B]$

B is T -closed set containing A^c by (5),

and A^c is T -closed set containing A .

Hence, $A^c = T$ closure of A

$$A^c = \bar{A}$$

(21)

ISS

Th.^m - 5. Homomorphism is an equivalence relative in the class of topological space.

Proof -

Homomorphism is reflexive - let (X, τ) is a t.s then

$(X, \tau) \approx (X, \tau)$ for, the identity mapping $I: X \rightarrow X$ with $I(x) = x \quad \forall x \in X$

Homomorphism is symmetric - let (X, τ_1) & (Y, τ_2) be two t.s.

$$= \begin{cases} f \text{ is one-one onto} \\ f \text{ is } \tau_1\text{-}\tau_2 \text{ continuous} \\ f^{-1} \text{ is } \tau_2\text{-}\tau_1 \text{ continuous.} \end{cases}$$

$$= \begin{cases} f^{-1} \text{ is one-on-onto} \\ (f^{-1})^{-1} \text{ is } \tau_1\text{-}\tau_2 \text{ continuous} \\ f^{-1} \text{ is } \tau_2\text{-}\tau_1 \text{ continuous} \end{cases}$$

$\Rightarrow f^{-1}$ is a homomorphism

$$(X, T_1) \cong (Y, T_2) \Rightarrow (Y, T_2) \cong (X, T_1)$$

Homomorphism is reflexive - let (X, T_1) , (Y, T_2) & (Z, T_3) be topological space then

$$f: (X, T_1) \rightarrow (Y, T_2)$$

$$g: (Y, T_2) \rightarrow (Z, T_3)$$

$$g \circ f: (X, T_1) \rightarrow (Z, T_3)$$

$$= \begin{cases} f \& g \text{ are one-one onto} \\ f \& g \text{ are continuous} \\ f^{-1} \& g^{-1} \text{ are continuous.} \end{cases}$$

$$= \begin{cases} g \circ f \text{ is one-one onto} \\ (g \circ f) \text{ is continuous} \\ (g \circ f)^{-1} \text{ is } = f^{-1} \circ g^{-1} = f^{-1} [g^{-1}] \text{ is continuous.} \end{cases}$$

$g \circ f$ is homomorphism.
Hence homomorphism is an equivalence relation.

21/19

* first Countable space - let (X, τ) be a Topological space, is said to be first countable space if for every $x \in X \exists$ a countable local base at x .

ex - discrete space $(X, \mathcal{D})$ is first countable

18

* Second Countable space - let (X, τ) be a Topological space $\mathcal{B}$ called second countable space if τ has a countable base.

or

Let (X, τ) be a topological space, The space X is said to be second countable space if $\exists$ a countable base for τ .

ex. $(\mathbb{R}, \mathcal{U})$ is second countable when $\mathcal{U}$ is usual To.

126 (19)

Theorem - Every second countable space is first countable.

proof. Let (X, τ) be a second countable space

i.e. $\mathcal{B} = \{B_1, B_2, \dots\}$ be a countable base for topology

Then we have to show that (X, T) is first countable.

let $x \in X$, N_x is nbd of system of

$$B(x) = \left\{ B : \begin{matrix} B \in \mathcal{B} \\ x \in B \end{matrix} \right\}$$

clearly $B(x)$ forms local base for x
 i.e. $B(x)$ forms base for nbd system N_x

i.e. $B(x)$ is countable.

$\Rightarrow B(x)$ is countable B

i.e. for any $x \in X$

$\exists$ countable local base at x .

Hence, (X, T) is first countable.

¹²⁸ Th^m - Every second countable space is separable.

proof - let (X, T) be a second countable space

i.e. $\mathcal{B} = \{ B_1, B_2, \dots \}$ be a countable base for topology T .

Then we have to show that X is separable

Here $\mathcal{B}$ is base for topology τ .

$$B_n \neq \emptyset \quad \forall n \in \mathbb{N}$$

$$\text{so, } x_n \in B_n \quad \forall n \in \mathbb{N}$$

Define, $A = \{x_n : n \in \mathbb{N}\}$

Since $\mathcal{B}$ is countable $\Rightarrow A = \{x_n\}$ is collection of $\mathcal{B}$.

Clearly, A is countable
 so, $A \subseteq X$ or $\bar{A} \subseteq X$ — (1)

Let $x \in X$, we get have two case.

(1) $x \in A$ & (2) $x \notin A$

(1) when $x \in A$
 $\Rightarrow x \in A \cup A' = \bar{A}$
 $\Rightarrow x \in \bar{A}$

so, if $x \in A \Rightarrow x \in \bar{A}$

(2) When $x \notin A$ & $x \in X$

let N be nbd of x

$\Rightarrow \exists G \in \mathcal{T}$, such that

$$x \in G \subseteq N$$

& $\mathcal{B}$ is a base for topology $\mathcal{T}$.
countable

$\Rightarrow \exists m \in \mathbb{N}$, $B_m \in \mathcal{B}$ such that

$$x \in B_m$$

$$\Rightarrow x \in B_m \subseteq G \subseteq N$$

$$\Rightarrow B_m \subseteq N$$

$$\Rightarrow x_m \in N$$

Now, $x \notin A \Rightarrow x \neq x_n \quad \forall n \in \mathbb{N}$

$$\text{so, } x_m \in A \cap (N - \{x\})$$

$$\Rightarrow A \cap (N - \{x\}) \neq \emptyset$$

Here N is arbitrary nbd of x .
so for every nbd N of x , we get

$$\boxed{A \cap (N - \{x\}) \neq \emptyset}$$

x is limit point of A

$$\Rightarrow x \in A'$$

$$\Rightarrow x \in A \cup A' = \bar{A}$$

$$\Rightarrow x \in \bar{A}$$

Thus for any $x \in X \Rightarrow x \in \bar{A}$

$$\Rightarrow X \subseteq \bar{A} \quad \text{--- (2)}$$

from eqⁿ (1) & (2)

$$\bar{A} \subset X$$

i.e. A is dense

Hence, A is countable dense subset of X

$\Rightarrow (X, T)$ is separable

143

Th^m $\Rightarrow$ A mapping f from a space X into another space Y is continuous if and only if

$$(20) (19) (23) \cdot f(\bar{A}) \subseteq \overline{f(A)} \quad \text{for every } A \subset X$$

proof - let (X, T) & $(Y, \mathcal{B})$ be two topological space.

Let f is continuous and $A \subset X$, then we have to show that

$$f(\bar{A}) \subset \overline{f(A)}$$

let $f(A) = B$
let $B \subset Y$, f is continuous.
then

$$\Rightarrow f^{-1}(B) \subset f^{-1}(\overline{B})$$

$$\Rightarrow f^{-1}[f(A)] \subset f^{-1}[\overline{f(A)}]$$

$$\Rightarrow f^{-1}[f(A)] \subset f^{-1}[\overline{f(A)}]$$

$$\boxed{\Rightarrow f(A) \subset \overline{f(A)}}$$

Converse - let $f: X \rightarrow Y$ is mapping
s.t.

$$f(A) \subset \overline{f(A)} \quad \text{for any } A \subset X.$$

Then we have to show that f is continuous.

let $B \subset Y$ be an arbitrary.

$$f^{-1}(B) \subset X$$

$$f\{f^{-1}(B)\} \subset \overline{f[f^{-1}(B)]}$$

$$f\{f^{-1}(B)\} \subset \overline{f[f^{-1}(B)]}$$

or $f^{-1}(B) \subset f^{-1}(\overline{B}) \quad \text{--- (1)}$

But $f^{-1}(B) \subset \overline{f^{-1}(B)}$ — (2)
 $\overline{B}$ is closed.

$$\overline{B} = B \quad \text{--- (3)}$$

$$\overline{f^{-1}(B)} \subset f^{-1}(B) \quad \text{--- (4)}$$

from (2) & (4)

$$f^{-1}(B) = \overline{f^{-1}(B)}$$

or

$$\overline{f^{-1}(B)} = f^{-1}(B)$$

$\Rightarrow f^{-1}(B)$: is closed set in X .

Hence B is closed in $Y \Rightarrow f^{-1}(B)$ is closed in X .

$\Rightarrow f$ is continuous.

156 Theorem:- 7 Show that a homomorphism image of a second countable space is second countable.

Proof - let (Y, τ_2) is a homomorphism image of second countable space (X, τ_1) and f is the corresponding homomorphism.

since (X, τ_1) is second countable base i.e $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ is countable base for τ_1 ,

When every $f[B_n]$ is T_2 -open as f is open

Any T_2 -open subset $H \subset Y \Rightarrow f^{-1}[H]$ is T -open as f is continuous.

$$f^{-1}[H] = \bigcup \{ B_\lambda : \lambda \in \Lambda \} \subset X$$

$$H = f \left[\bigcup B_\lambda : \lambda \in \Lambda \right]$$

$$= \bigcup \{ f[B_\lambda] : \lambda \in \Lambda \}$$

$$= \text{Union of members of } \{ f[B_n] : n \in \mathbb{N} \}$$

$$= B = \{ f[B_n] : n \in \mathbb{N} \} \text{ is countable}$$

base for T_2

Hence (Y, T_2) is a second countable.

A topological space

★ Separable space:- (X, τ) is separable iff X contains a countable dense subset.

$A \subset X$

$$\bar{A} = X$$

countable.

$(\mathbb{R}, \tau)$

example -

$(\mathbb{R}, \tau_S)$ is separable

$\phi: \mathbb{Q} \rightarrow \mathbb{R}$

$$\bar{\phi} = \mathbb{R}$$

ϕ countable
dense

$$\bar{\phi} = \phi \cup \phi'$$

$\downarrow \mathbb{R}$

ex. 2 (X, τ_c) is separable $\Leftrightarrow X$ is countable

ex. 3 (X, τ_c) is separable. (cofinite Topology)
 $\checkmark$ countable

if X is not countable,

X contains a (non-finite) countable subset A .

$$\tau_c = \{ \phi, U : U^c \text{ is finite} \}$$

τ_c -closed sets are finite or X

$\bar{A} = X \Rightarrow (X, \tau_c)$ is separable.

* Homeomorphism/ Topologically Equivalent:-

$f: X \rightarrow Y$ is said to be homeomorphism if

- (i) f is bijective. (1-1 + onto).
- (ii) f is continuous. (every open set has inverse open set)
- (iii) f^{-1} is continuous. (every open set is also)

$$(f^{-1}: Y \rightarrow X) \quad (f \text{ is open})$$

$$X \cong Y$$

$$f(u) \sim u$$

ex. $f: (0, 1) \rightarrow (1, \infty)$

$$f(x) = \frac{1}{x} \Rightarrow f(x) = f(y) \Rightarrow x = y$$

$$\downarrow$$

$$\frac{1}{x} = \frac{1}{y} \Rightarrow y = x$$

$$\forall x \in (0, 1)$$

f is injective

Surjective

$\forall b \in (1, \infty) \exists a \in (0, 1)$ such that $f(a) = b$

$b \in (1, \infty)$, $b \geq 1$, $0 < \frac{1}{b} < 1$ let $a = \frac{1}{b}$

$$f(a) = \frac{1}{a} = \frac{1}{\frac{1}{b}} = b \Rightarrow f(a) = b$$

(ii) - Continuous

Theorem:- A continuous image of separable space is separable.

or
property of a space being separable is topological property.

$\rightarrow (X, \tau) \rightarrow$ separable, $f: X \rightarrow Y$, f is continuous + onto

(Y, ν) is continuous image of (X, τ)

A be countable dense subset of X ($A = \bar{A}$)
 $\Rightarrow f(A)$ is surely countable.

Now $\bar{A} = X \Rightarrow Y = f[X]$
 $= f[\bar{A}]$

f is cts $\Rightarrow f(\bar{A}) \subset \overline{f(A)}$

$$Y \subset \overline{f(A)} \rightarrow \overline{f(A)} \subset Y$$

But $\overline{f(A)} = Y$

Hence Y is separable.

(23)
Thm $\rightarrow$ A metric space is separable $\Leftrightarrow$ it is second countable.

Second countable $\Rightarrow$ separable
 $\downarrow$ prove

$(X, d) \rightarrow$ separable metric space

$A = \{x_n : n \in \mathbb{N}\}$ be countable dense subset of X i.e. $\bar{A} = X$

We claim $B = \{S(x_n, 1/m) : n \in \mathbb{N}, m \in \mathbb{N}\}$ is countable Base for $\mathcal{T}$.

$\Rightarrow$ let $x \in X$, G be open set s.t. $x \in G$

$\Rightarrow \exists \epsilon > 0$ s.t. $S(x, \epsilon) \subset G$

Choose $m_0 \in \mathbb{N}$ s.t. $1/m_0 < \frac{\epsilon}{2}$

since $A = X \Rightarrow x$ is Adherent point of A

$\Rightarrow S(x, 1/m_0)$ must contain point of A

let say $x_i \in S(x, 1/m_0) \cap A$

$$\Rightarrow d(x, x_i) < 1/m_0$$

$$\Rightarrow x \in S(x_i, 1/m_0)$$

To show that $S(x_i, 1/m_0) \subset S(x, \epsilon)$

$$\text{let } y \in S(x_i, 1/m_0) \Rightarrow d(x_i, y) < 1/m_0$$

$$d(x, y) \leq d(x, x_i) + d(x_i, y)$$

$$< \frac{1}{m_0} + \frac{1}{m_0} = \epsilon$$

$$\Rightarrow y \in S(x, \epsilon)$$

$$\therefore S(x_i, 1/m_0) \subset S(x, \epsilon) \subset G$$

To every $x \in X$ & every open set G containing x , $\exists S(x_i, 1/m_0) \subset G$ & contained in G

$\Rightarrow B$ is countable Base

Thm- Every open continuous image of a second co. sp. is sec. co.

Second countability is topological property.
 let

$f: X \rightarrow Y$, Open continuous & onto map
 (Y, V) is image of (X, T) .

pr.
 (X, T) is second co., then (Y, V) is also sec. co.

let $\mathcal{B}$ be countable Base for T

f is open $\Rightarrow f[B]$ is v -open when $B \in \mathcal{B}$

let G be any v -open subset of $Y \Rightarrow$
 $f^{-1}(G)$ is T -open $\because f$ is

$\therefore f$ is etc.

$$f^{-1}(G) = \cup \{B : B \in \mathcal{B}\} \Rightarrow f(f^{-1}(G)) =$$

$$f[\cup \{B : B \in \mathcal{B}\}]$$

$$G = \cup [f(B) : B \in \mathcal{B}] \Rightarrow \{f[B] : B \in \mathcal{B}\}$$

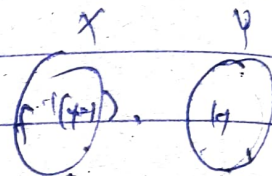
forms countable Base of V .

Continuous function - let (X, τ) , & (Y, ν) topological space

function $f: X \rightarrow Y$ is cts at x_0 iff f to every ν -nhd M of $f(x_0)$ $\exists$ τ -nhd N of x_0 s.t. $f(N) \subset M$

$\rightarrow$ iff for every ν -open set M containing $f(x_0)$ $\exists$ τ -open set N containing x_0 s.t. $f(N) \subset M$

Thm $\checkmark$ $f: X \rightarrow Y$ is continuous $\Leftrightarrow$ the inverse image under f of every open set is open in X .



$\Rightarrow$ let f be continuous.

$H \subset Y$, H is open

τ -s $\rightarrow f^{-1}(H)$ is open

$f^{-1}(H) = \emptyset$, Nothing to prove.

if $f^{-1}(H) \neq \emptyset$, $x \in f^{-1}(H)$

$$f(x) \in H$$

Since f is cts $\exists G_x$ s.t. $f(G_x) \subset H$

$$\Rightarrow x \in G_x \subset f^{-1}(H)$$

$\Rightarrow f^{-1}(H)$ is nbd of each of its points

$\Rightarrow f^{-1}(H)$ is open

$\Leftarrow f^{-1}(H)$ is open in X for every open set H in Y

T.S. - f is cts.

Let H be open in Y s.t.

$$f(x) \in H \Rightarrow x \in f^{-1}(H)$$

since $f^{-1}(H)$ is open

It is nbd of $x \Rightarrow \exists$ open set G s.t. $x \in G \subset f^{-1}(H)$

$$f(G) \subset H$$

$\Rightarrow f$ is cts.

★ Open function :- Let $f: X \rightarrow Y$ be a function, on the then f is an open function or open mapping if for each open set U in X $f(U)$ is open in Y .

The function f is a closed function if for each closed set C in X $f(C)$ is closed in Y .

• Sequential continuity at a point :-
 A funⁿ $f: X \rightarrow Y$ is sequentially continuous at $p \in X$ iff for every sequence $\langle a_n \rangle$ in X converging to $p \in X$ the sequence $\langle f(a_n) \rangle$ in Y converges to $f(p) \in Y$ i.e.

$$a_n \rightarrow p \text{ implies } f(a_n) \rightarrow f(p)$$

Lindelof Space

(X, τ) is said to be Lindelof space iff every τ -open cover of X has a countable subcover.

Compact space $\Rightarrow$ Lindelof space
 $\searrow$ finite subcover

But converse is not true

$\searrow$ countable subcover

$(X, \mathcal{D})$, X -infinite countable set is Lindelof but not compact, Subcover - Singleton set - Subcover

Ex - $(\mathbb{R}, \mathcal{U})$ is $\Rightarrow$ Lindelof.

The Lindelof theorem:-

Every second countable space is Lindelof space.

Let $(X, \tau) \rightarrow$ second countable space

$B \rightarrow$ countable Base for τ

Let $\mathcal{C}$ be an open cover of X

each member of $\mathcal{C}$ can be expressed as union of members of B .

Let $\mathcal{D}$ be collection of all those members of B which are required for expressed members of $\mathcal{C}$ as union of members of B .

$\Rightarrow \mathcal{D}$ is countable subcollection of countable Base B .

$\Rightarrow \mathcal{D}$ is countable open cover of X .

for each $D_i \in \mathcal{D}$ we choose $C_i \in \mathcal{C}$ s.t.
 $D_i \subset C_i$

$\Rightarrow$ the collection of C_i is also countable open cover of X .

$\Rightarrow \mathcal{C}$ has countable subcover.

Hence X is Lindelof space.

Second countability is a hereditary property

proof - (X, τ) is second countable space

$$\mathcal{B} = \{B_1, B_2, B_3, \dots\} \text{ for } (X, \tau)$$

$$\forall x \in X, \tau_x = \{A \cap U : A \in \mathcal{B}\}$$

Subspace

$$\mathcal{B}^* = \{B_1 \cap Y, B_2 \cap Y, \dots\} (Y, \tau_Y)$$

$$\mathcal{B} \subseteq \tau, B_i \in \tau$$

$$\mathcal{B} \cap \tau \in \tau_Y$$

$$\Rightarrow \mathcal{B}^* \subseteq \tau_Y$$

$$y \in Y, N - \tau_Y$$

$$y \in G \subseteq \tau_Y, G \in \tau_Y$$

$$G = A \cap Y, A \in \tau$$

$$y \in A \cap Y \subseteq N \Rightarrow y \in A, A \in \tau$$

$$y \in B_i \subseteq A, B_i \in \mathcal{B}$$

$$y \in Y$$

$$y \in B_i \cap Y \subseteq B_i \in A \cap Y \subseteq N$$

$$y \in B_i \cap Y \subseteq N, B_i \cap Y \in \mathcal{B}^*$$

$$\Rightarrow \mathcal{B}^* \text{ is base for } (Y, \tau_Y)$$

Chapter - 8.

Compactness.

Finite cover :- If number of elements in C is finite then C is called finite cover.

Sub-cover - Let (X, T) be a topological space and C is its covers and any $C^* \subseteq C$ if C^* forms cover of X then C^* is called subcover of C .

Cover - Let (X, T) be a topological space. A collection C of subsets of X i.e. $C = \{C_i : i \in I\}$, Here $C_i \subseteq X \forall i \in I$. If $X = \bigcup_{i \in I} C_i$

Then C is called covering of X .

Open cover - If every element of C are T -open then C is said to be open cover.

Compact - Let (X, T) be a topological space and A is subset of X . $A \subseteq X$ is said to be compact iff every open covering of X has a finite subcover.

201 July 19
 Thend - Closed subsets of compact sets are compact.

proof - let (X, τ) be a topological space and let Y be its compact subset.

Then we have to show that closed subset of Y is compact.

Let F be closed subset of Y .
 i.e. $F \subseteq Y \subseteq X$ & F is τ -closed.

let $\mathcal{C} = \{G_\alpha : \alpha \in \Lambda\}$ be an open cover of F .

$$\text{i.e. } F \subseteq \bigcup_{\alpha \in \Lambda} G_\alpha$$

G_α is τ -open in $X \Rightarrow \forall \alpha \in \Lambda$
 and F is τ -closed.

Then, $X - F = F^c$ is τ -open

$$\left(\bigcup_{\alpha \in \Lambda} G_\alpha \right) \cup F^c = X$$

$\Rightarrow \mathcal{D} = \mathcal{C} \cup \{F^c\}$ form an open cover of X .

$\mathcal{D}$ is open cover of Y .

$\mathcal{D}$ has a finite subcollection

$$\mathcal{D}^* = \{ G_{n_i} \mid i=1, 2, \dots, k \} \cup \{ F^c \} \subseteq \mathcal{D}$$

Such that $\mathcal{D}^*$ cover Y .

$$\Rightarrow Y \subseteq \left(\bigcup_{i=1}^k G_{n_i} \right) \cup F^c$$

$$F \cap Y \subseteq F \cap \left(\bigcup_{i=1}^k G_{n_i} \right) \cup F^c$$

$$\Rightarrow F \subseteq \bigcup_{i=1}^k G_{n_i}$$

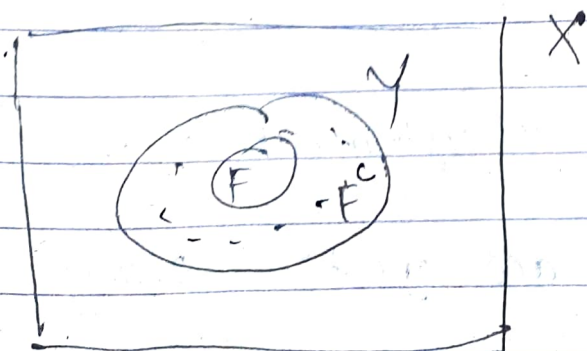
$$\text{Let } \mathcal{C}^* = \{ G_{n_i} \mid i=1, 2, \dots, k \}$$

Clearly, $\mathcal{C}^*$ form an open cover of F &
 $\mathcal{C}^* \subseteq \mathcal{C}$

i.e. $\mathcal{C}^*$ is finite subcover of $\mathcal{C}$.

Thus, every open cover of F has a finite subcover.

Hence F is compact.



* Hausdorff space (T_2 -space)

Let (X, T) be a topological space, X is said to be T_2 -space or Hausdorff space if for any distinct element x & y of X $\exists$ two distinct T -open set G and H .

Such that $x \in G$ & $y \in H$
 $\Rightarrow G \cap H = \emptyset$

Pg. 202 ✓

Thm^m-4 Every Compact subset of A of Hausdorff space X is closed.

Proof - Let (X, T) be a Hausdorff space

Then we have to show that compact subset of Hausdorff space is closed.

Let A be compact subset of Hausdorff space

$$\text{i.e. } A \subseteq X$$

Here

$A^c = X - A$, is complement of A .

$$A \cap A^c = \emptyset$$

i.e. for any $x \in A$ and $y \in A^c$, $x \neq y$

Let $y \in A^c$ be arbitrary

$\Rightarrow$ for any $x \in A$

$x, y \in X$ where $x \neq y$

$\exists$ two distinct open set N_x & M_x at x such that

$\Rightarrow x \in N_x$ & $y \in M_x$
 such that $N_x \cap M_x = \emptyset$

$$\Rightarrow \bigcup_{x \in A} \{x\} \subseteq \bigcup_{x \in A} N_x$$

$$\Rightarrow A \subseteq \bigcup_{x \in A} N_x$$

Clearly $C = \{N_x : x \in A\}$ forms an open cover of A & A is compact.

$\exists C^*$ a finite subcover of C such that

C^* forms cover of A .

$$\text{i.e. } C^* = \{N_{x_i} ; i = 1, \dots, k\}$$

$$\& A \subseteq \bigcup_{i=1}^k N_{x_i}$$

So, corresponding to each N_{x_i} we get M_{x_i}

$$N_{x_i} \cap M_{x_i} = \emptyset$$

Then M is open nbd of y , then
we have to show that
 $\Rightarrow M \cap N = \emptyset$

$$x \in N \Rightarrow x \in N_{x_i} \quad \forall i = 1, 2, \dots, k$$

$$\Rightarrow x \notin M_{x_i}$$

$$\Rightarrow x \notin M$$

$$\Rightarrow N \cap M = \emptyset \quad \& \text{ Since } A \subseteq N$$

$$\Rightarrow A \cap M = \emptyset$$

$$\Rightarrow M \subseteq A^c$$

$\Rightarrow A^c$ contains a nbd of each of its point

$\Rightarrow A^c$ is open

$\Rightarrow A$ is closed.

PG 203 19

Theorem 6) A continuous image of a compact set is compact.

proof - Let (X, τ) & (Y, δ) be compact topological space.
and

Let $f: X \rightarrow Y$ be a continuous function

Then we have to show that $f(X)$ is compact.

let $C = \{G_\alpha : \alpha \in \mathbb{N}\}$ be any compact open cover of $f(X)$.

Since, f is continuous,

$\Rightarrow f^{-1}[G_\alpha]$ is open in X .

The collection $\{f^{-1}[G_\alpha] : \alpha \in \mathbb{N}\}$ then forms an open cover of X .

Since, X is compact then there exist finitely subcollection such that

$$X = f^{-1}[G_{\alpha_1}] \cup f^{-1}[G_{\alpha_2}] \cup \dots \cup f^{-1}[G_{\alpha_n}]$$

$$= f^{-1}[G_{\alpha_1} \cup G_{\alpha_2} \cup \dots \cup G_{\alpha_n}]$$

$$f(X) = G_{\alpha_1} \cup G_{\alpha_2} \cup \dots \cup G_{\alpha_n}$$

So, $f(X)$ have a finite sub collection

$\Rightarrow f(X)$ is compact.

P-3-204

★ finite intersection property (F.I.P.) -

A family $A^* = \{A_i\}$ of sets is said to be finite intersection property if the intersection of members of every finite sub family say $(A^*)_0 = \{A_{i_1}, A_{i_2}, \dots, A_{i_n}\}$ is non-empty.

$$\text{i.e. } A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_n} \neq \emptyset$$

fixed - A class of sets is said to be fixed if it has non-empty intersection.

P-3-205 22
Th^m -

A topological space X is compact if every collection of closed subsets of X with the finite intersection property is fixed, i.e. has a non-empty intersection.

proof - let (X, τ) be a topological space and let $\mathcal{C}$ be compact.

and let $\mathcal{F} = \{F_d : d \in \mathbb{N}\}$ be a collection of closed subsets of X with F.I.P.

Suppose if possible.

$\cap \{F_\alpha : \alpha \in \Lambda\}$ be a collection of closed subsets of X with F.I.P.

$$\cap \{F_\alpha : \alpha \in \Lambda\} = \phi$$

then,

$$[\cap \{F_\alpha : \alpha \in \Lambda\}]' = \phi'$$

$$[\cap \{F_\alpha : \alpha \in \Lambda\}]' = X$$

$$\cup \{F'_\alpha : \alpha \in \Lambda\} = X \quad \left\{ \because \text{by De-morgan's} \right\}$$

$\Rightarrow \{F'_\alpha : \alpha \in \Lambda\}$ is open cover of X

since, F_α are closed,

since X is compact i.e. every open cover has finite subcover.

$$\cup \{F'_{\alpha_i} : i = 1, 2, \dots, n\} = X$$

$$[\cap \{F_{\alpha_i} : i = 1, 2, \dots, n\}]' = X \quad \left\{ \because \text{by de-morgan} \right\}$$

Converse:— Let every collection of $E_i \subset X$ subsets of X with FIP have a non-empty intersection.

and let $C = \{G_\alpha : \alpha \in \Lambda\}$ be an open cover of X .

so that $X = \bigcup \{G_\alpha : \alpha \in \Lambda\}$.

On applying complement, we get

$$X' = \left[\bigcup \{G_\alpha : \alpha \in \Lambda\} \right]'$$

$$\phi = \bigcap \{G_\alpha' : \alpha \in \Lambda\} \quad \text{by de-morgan}$$

Thus, $\{G_\alpha' : \alpha \in \Lambda\}$ is a collection of closed sets with empty intersection & so by cover hypothesis this collection does not have the F.I.P.

Hence $\exists$ a finite no. of sets
i.e. $\{G_{\alpha_i}' : i = 1, 2, \dots, n\}$

$$\bigcap \{G_{\alpha_i}' : i = 1, 2, \dots, n\} = \phi$$

$$\left[\bigcap \{G_{\alpha_i}' : i = 1, \dots, n\} \right]' = \phi'$$

$$\bigcup \{G_{\alpha_i} : i = 1, \dots, n\} = X$$

Hence, X is compact.

Separation axioms

T_0 -space - A topological space (X, τ) is said to be T_0 -space or T_0 -topology iff given any distinct point x and y of X $\exists$ a nbd of one of them not containing other.

$$\text{i.e. } \forall x, y \in X, x \neq y$$

either $\exists$ open set $G \in \tau$ s.t. $x \in G, y \notin G$
 or $\exists$ open set $H \in \tau$ s.t. $x \notin H, y \in H$
 ex. Every Discrete space is T_0 space.

T_1 space - A topological space (X, τ) is said to be T_1 -space iff every distinct point x, y of X , $\exists$ ^{two} open sets G & H of X such that $x \in G, y \notin G$ & $y \in H, x \notin H$.

$$\Rightarrow T_1 \Rightarrow T_0$$

(R.V.) is $T_1 \rightarrow x, y \in \mathbb{R}, y > x, |x - y| = k$.

227

Thm - A topological space (X, τ) is a T_1 -space
proof - Let (X, τ) be a topological space

$\{x\}$ is closed $\Rightarrow x \in X$ is closed

$X - \{x\}$ is open $\Rightarrow$ its complement is open

$x \notin X - \{x\}, y \in X - \{x\} (y \neq x)$ has nbd which does not contain x .

$\{x\}$ is closed $\Rightarrow$ every singleton subset $\{x\}$ of X is T -closed.

$\Rightarrow (X, \tau)$ is T_1 -space

Conversely, let (X, τ) is T_1 -space

So Every $x, y \in X$, $x \neq y$
 $\exists$ open set G & H of X ,
 s.t.

$$x \in G, y \notin G \text{ \& } y \in H, x \notin H$$

$y \notin G$; $x \in G$ & G is open set of X

$\Rightarrow$ so its complement is closed in X
 $\Rightarrow \{x\} \in X$ is closed.

Th^m-2. Every subspace of T_1 -space is a T_1 -space & hence the property

proof - let (X, τ) be a T_1 -space & let (Y, δ) be a subspace of (X, τ)

let y_1, y_2 be two distinct points of Y

$$y_1, y_2 \in Y$$

since $Y \subset X$

$$\Rightarrow y_1, y_2 \in X$$

$\Rightarrow \exists$ an open sets $[X \text{ is } T_1\text{-space}]$
 $G \text{ \& } H$ s.t.

and $y_1 \in G$, $y_2 \notin G$
 $y_2 \in H$, $y_1 \notin H$

$$y_1 \in G \cap Y = G_1 \quad \& \quad y_2 \in H \cap Y = H_1$$

Then $G_1 = G \cap Y$ & $H_1 = H \cap Y$ are δ -open sets.
 s.t.

$$y_1 \in G_1, \text{ but } y_2 \notin G_1,$$

$$y_2 \in H_1, \text{ but } y_1 \notin H_1,$$

$\Rightarrow (Y, \delta)$ is a T_1 -space.

Thm-3 The homeomorphism image of T_1 -space is T_1 -space.

proof - Let (X, τ_1) be a topological space.

and let (Y, τ_2) be its space homeomorphism
 let f be the corresponding homeomorphism

i.e. $f: X \rightarrow Y$ then:

- (i) f is one-one onto $\Rightarrow f(X) = Y$
- (ii) f is continuous.
- (iii) f^{-1} is continuous.

but (X, τ_1) is T_1 -space

$\Rightarrow$ given distinct point $x, y \in X$
 $\exists G, H \in \mathcal{T}_1$ s.t.
 $x \in G$ & $y \notin G$
 $y \in H$ & $x \notin H$.

$\Rightarrow f(x), f(y) \in Y$ s.t. $f(x) \neq f(y)$

$\Rightarrow \exists f(G), f(H) \in \mathcal{T}_2$ s.t.

$f(x) \in f(G)$ but $f(y) \notin f(G)$
 and $f(y) \in f(H)$ but $f(x) \notin f(H)$

$\Rightarrow f(y) \in f(H)$ but $f(x) \notin f(H)$

$\Rightarrow (X, \mathcal{T}_2)$ is $\mathcal{T}_1$ -space
 Hausdorff / separated.

Pg 242

$\mathcal{T}_2$ -space - A topological space $(X, \mathcal{T})$ is said to be $\mathcal{T}_2$ -space or Hausdorff space iff for every pair of distinct point x, y of X $\exists$ a disjoint nbd of x & y .

i.e. $\exists$ nbd N of x & nbd M of y
 s.t.

$$N \cap M = \emptyset$$

ex. Every Discrete space is $\mathcal{T}_2$

948

Th^m-1. Every metric space is a T_2 -space.

proof: Let (X, d) be metric space
 Let τ be metric topology on X
 corresponding to metric d .

To prove that (X, τ) is a T_2 -space
 let $x, y \in X$ s.t. $x \neq y$.

Let $d(x, y) = \varepsilon$ so that $\varepsilon > 0$

Then the open spheres $S(x, \frac{\varepsilon}{2})$ &
 $S(y, \frac{\varepsilon}{2})$ are distinct nbd of x & y .

$$\Rightarrow S(x, \frac{\varepsilon}{2}) \cap S(y, \frac{\varepsilon}{2}) = \emptyset$$

$\Rightarrow X$ is T_2 -space.

244

Th^m-20. Every convergent sequence in a Hausdorff space has a unique limit.

proof - Let (X, τ) be a Hausdorff space
 & let $\langle x_n \rangle = \{x_1, x_2, \dots, x_n, \dots\}$ be
 a convergent sequence in X .

Then we have to show that limit of
 this sequence is unique.

Suppose if possible $\langle x_n \rangle$ converges to two distinct points x & y .

Since X is Hausdorff space.

$$\Rightarrow x, y \in X, x \neq y$$

$$\Rightarrow \exists \text{ open set } G \text{ \& } H \text{ s.t.}$$

$$\Rightarrow x \in G, y \in H$$

$$\Rightarrow G \cap H = \emptyset$$

and $\langle x_n \rangle \rightarrow x$

$$\Rightarrow \exists n_0 \in \mathbb{N} \text{ s.t. } n \geq n_0 \Rightarrow x_n \in G$$

and $\langle x_n \rangle \rightarrow y$

$$\Rightarrow \exists m_0 \in \mathbb{N} \text{ s.t. } n \geq m_0 \Rightarrow x_n \in H,$$

let $N = \max\{n_0, m_0\}$. Then

$$x_n \in G \text{ \& } x_n \in H$$

$$\Rightarrow n_0 \in G \cap H$$

$$\Rightarrow G \cap H \neq \emptyset$$

which contradicts our hypothesis that $G \cap H = \emptyset$

hence, the limit of this sequence is unique.

245 ✓

Thm-5 Every subspace of Hausdorff space is Hausdorff space & hence the property is hereditary.

proof - let (X, τ) be Hausdorff space and let (Y, δ) be subspace of Hausdorff space.

i.e. $Y \subset X$ such that $Y \neq \emptyset$

let $x, y \in Y \Rightarrow x, y \in X, x \neq y$

Since X is Hausdorff space

$\Rightarrow \exists$ an open set G & H of X & y s.t.

$$x \in G \text{ & } y \in H \Rightarrow G \cap H = \emptyset$$

$$\text{so, } x \in Y \text{ & } x \in G \Rightarrow x \in G \cap Y$$

$$\text{and } y \in H \text{ & } y \in Y \Rightarrow y \in H \cap Y$$

So these are δ -open set

$$\text{since } G \cap H = \emptyset$$

$$\text{we have, } (Y \cap G) \cap (Y \cap H) = Y \cap (G \cap H)$$

$$= Y \cap \emptyset$$

$$= \emptyset$$

$$\Rightarrow (Y \cap G) \cap (Y \cap H) = \emptyset$$

Thus $G \cap Y$ & $H \cap Y$ are disjoint τ -open nbds of x & y respectively,

Hence, (Y, δ) is Hausdorff space.

T_4 -space or normal - A topological space (X, T) is said to be normal, iff every pair L, M of point T -closed subset of X $\exists$ an T -open set G and H such that $L \subset G$ and $M \subset H \Rightarrow G \cap H = \emptyset$

Unit IV Separated set

Let (X, T) be a topological space Two non-empty subsets A and B of X are said to be T -separated iff

$$\{ A \cap \bar{B} = \emptyset \text{ and } \bar{A} \cap B = \emptyset \}$$

Thus A and B are separated $\Leftrightarrow A$ and B are disjoint and neither of them contains a limit point of other.

$$A \cap \bar{B} = \emptyset \Leftrightarrow A \cap [\bar{B} \cap \bigcup D(B)] = \emptyset$$

$$\Leftrightarrow (A \cap B) \cup (A \cap D(B)) = \emptyset \Leftrightarrow A \cap D(B) = \emptyset$$

$\Leftrightarrow A$ contains no limit point of B .

$$\bar{A} = A \cup D(A)$$

$$\text{Ex - } A = (2, 3)$$

$$\bar{A} = [2, 3]$$

$$B = (3, 4)$$

$$\bar{B} = [3, 4]$$

$$A \cap B = \emptyset$$

$$\bar{A} \cap B = \emptyset$$

$$A \cap \bar{B} = \emptyset$$

Theo 2 A closed subspace of a Normal space is normal.

proof- let (X, τ) be a topological space which is normal

and let $(Y, \mathcal{U})$ be a closed subspace of (X, τ)
i.e. $Y \subset X$

we have to prove Y is normal.

Let $F_1, F_2 \subset Y$ be disjoint sets which are closed in Y .

and Y is closed in X , $F \subset Y \subset X$

$$F_1, F_2 \subset X$$

$\therefore F_1$ and F_2 are disjoint closed sets of X since X is normal

$\exists G_1, G_2 \in \tau$ such that

$$F_1 \subset G_1, F_2 \subset G_2$$

$$G_1 \cap G_2 = \emptyset$$

since $F_1 \subset Y$

$$\text{Now, } F_1 \subset G_1 \Rightarrow F_1 \cap F_1 \subset G_1 \cap Y$$

$$\Rightarrow F_1 \subset G_1 \cap Y$$

similarly $F_2 \subset Y$ &

$$F_2 \subset G_2 \Rightarrow F_2 \cap F_2 \subset G_2 \cap Y$$

$$\Rightarrow F_2 \subset G_2 \cap Y$$

By definition of relative topology.

$$G_1, G_2 \in \mathcal{T} \Rightarrow Y \cap G_1, Y \cap G_2 \in \mathcal{U}$$

$$\begin{aligned}(G_1 \cap Y) \cap (G_2 \cap Y) &= (Y \cap Y) \cap (G_1 \cap G_2) \\ &= Y \cap \phi \\ &= \phi\end{aligned}$$

Therefore given pair of disjoint closed sets F_1, F_2 in Y , $\exists$ disjoint sets.

$$G_1 \cap Y, G_2 \cap Y \in \mathcal{U} \text{ s.t.}$$

$$F_1 \subset G_1 \cap Y, \text{ \& } F_2 \subset G_2 \cap Y$$

$$(G_1 \cap Y) \cap (G_2 \cap Y) = \phi$$

$\Rightarrow (Y, \mathcal{U})$ is normal space.

T_3 -space - (Regular space $\uparrow$ T_1)

(X, T) is $\downarrow$ regular iff for every T -closed set F and every point $p \notin F \exists$ T -open sets G & H s.t. $F \subset G$, $p \in H$ & $G \cap H = \emptyset$

T_3 space - Regular T_1 space is called T_3 space.

$T_1 = x, y (x \neq y) \in X, \exists G \& H$ (open set) s.t.
 $x \in G, y \notin G, y \in H, x \notin H$

$X = \{a, b, c\}$, $T = \{\emptyset, \{a\}, \{b, c\}, X\}$ regular but not T_3 closed subsets are
 $X, \{b, c\}, \{a\}, \emptyset$

Completely Normal space & T_5 -space

$\rightarrow$ A topological space X is said to be completely normal iff any two separated subsets A and B of $X \exists$ open set G and H s.t.

$A \subset G$, $B \subset H$ and $G \cap H = \emptyset$

T_5 space - A completely normal space which is T_1 is called T_5 -space.

pg. 169.

* ⁽²²⁾ Separated sets - Let (X, T) be a topological space. Two non-empty subsets of A and B are said to be separated sets iff -

$$A \cap \bar{B} = \phi \text{ and } \bar{A} \cap B = \phi$$

These two conditions are equivalent to the single condition

$$(A \cap B) \cup (\bar{A} \cap \bar{B}) = \phi$$

pg. 170

Theorem 2) Let A and B are separated subsets of a space X & $C \subset A$ and $D \subset B$ then C and D are also separated.

proof - Given that A & B are separated subsets of X ,
and $C \subset A$ and $D \subset B$.

Since, A & B are separated then

$$\therefore A \cap \bar{B} = \phi \text{ and } \bar{A} \cap B = \phi$$

These two conditions are equivalent to

Since, $C \subset A$ & $D \subset B$

$$\Rightarrow \bar{C} \subset \bar{A} \text{ & } \bar{D} \subset \bar{B}$$

$\Rightarrow C \cap \bar{D} = \emptyset$ and $\bar{C} \cap D = \emptyset$
Hence, C and D are separated.

Pg-172 (19)

7.2 Connected and disconnected sets -

Let (X, τ) be topological space. X is said to be disconnected if $\exists$ two non-empty sets A and $B \subset X$ such that

$$A \cup B = X, \quad A \cap \bar{B} = \emptyset \quad \text{and} \quad \bar{A} \cap B = \emptyset$$

The space (X, τ) is connected iff it is not disconnected.

Ex. 1 If $X = \{p, q, r\}$, $\tau = \{\emptyset, X, \{p, r\}, \{q\}\}$
Then show that topological space (X, τ) is disconnected.

Solⁿ - τ -open subsets are $\emptyset, X, \{p, r\}, \{q\}$

τ -closed subsets are $X - \emptyset, X - X, X - \{p, r\}, X - \{q\}$

$$\Rightarrow X, \emptyset, \{q\}, \{p, r\}$$

so, $\{q\}$ is non-empty proper subset of X which is both T -open & T -closed.

$\Rightarrow (X, T)$ is disconnected.

(18)

✓ Ex-2 Show that (X, T) is connected space if $X = \{p, q, r, s\}$ and $T = \{X, \phi, \{p, q\}\}$

Solⁿ - T -open subsets are $X, \phi, \{p, q\}$

T -closed sets are $X-X, X-\phi, X-\{p, q\}$

$\Rightarrow \phi, X, \{r, s\}$

Thus $\nexists$ no proper subset of X which is both open & closed.

Clearly (X, T) is not disconnected.

Hence (X, T) is connected.

(19)

P-9-175.

✓ Ex-7 The set of real no. with usual topology is connected.

Solⁿ - Let U be usual topology on $\mathbb{R}$.

Then we have to show that $(\mathbb{R}, U)$ is connected.

Supposed the contrary

that- $(\mathbb{R}, U)$ is disconnected then $\exists$ a disconnected on

$G \cup H$ for R

$$G \neq \emptyset, H \neq \emptyset$$

$$\Rightarrow \exists p \in G, q \in H$$

$$\Rightarrow p \neq q \text{ for } G \cap H = \emptyset$$

In particular, let $p < q$

$$p \in G, q \in H, \Rightarrow p, q \in R \text{ for } G, H \subset R$$

$$\Rightarrow [p, q] \subset R = G \cup H$$

$$\Rightarrow [p, q] \subset G \cup H$$

$$\text{any } t \in [p, q]$$

$$t \in [p, q] \subset G \cup H$$

$$\Rightarrow t \in G \text{ or } t \in H$$

Define $\tau = \sup \{ t : t \in \{p, q\} \cap G \}$

since $\tau = \sup [p, q]$ and G both are closed in $\mathbb{Q}$ hence by definition of τ .

$$\tau = [p, q] \text{ \& } \tau \in G$$

$$p \leq \tau \leq q$$

$$x \in G, G \cap H = \emptyset \Rightarrow x \notin H \Rightarrow x \neq q$$

$$\Rightarrow p \leq x \leq q, x \neq q \Rightarrow p \leq x < q$$

It is clear that,

$$x + \varepsilon \in H \quad \forall \varepsilon > 0 \text{ s.t. } x + \varepsilon \leq q$$

H is closed in $\mathbb{R}$ Hence $x \in H$

$$\Rightarrow x \in G, x \in H \Rightarrow x \in G \cap H$$

Contrary to the fact that $G \cap H = \emptyset$
 Hence the required result follows.

pg 178. (22)

Th.^m - 8 The closure of connected set is connected.
 (21)

Proof - Let Y be connected subset of a topological space (X, τ)

To prove that $\bar{Y}$ is connected.

(opposite)

Suppose contrary, that $\bar{Y}$ is disconnected.

i.e. $\exists$ non empty sets $A, B \subseteq \bar{Y}$ s.t.
 $\bar{A} \cap \bar{B} = \emptyset$

$$A \cap B = \phi, \quad \bar{Y} = A \cup B$$

$$A \cup B = Y \supset Y, \quad Y \text{ is connected} \quad \text{F: } Y = \bar{Y}$$

$$\Rightarrow Y \subset A \cup B, \quad Y \text{ is connected,}$$

$$\Rightarrow Y \subset A \text{ or } Y \subset B$$

$$Y \subset A \Rightarrow \bar{Y} \subset \bar{A}$$

$$\Rightarrow \bar{Y} \cap \bar{B} \subset \bar{A} \cap B$$

$$\Rightarrow \bar{Y} \cap B \subset \phi$$

$$\Rightarrow \bar{Y} \cap B = \phi \quad \text{--- (1)}$$

$$\bar{Y} = A \cup B \Rightarrow B \subset \bar{Y} \Rightarrow \bar{Y} \cap B = B \quad \text{--- (2)}$$

from (1) or (2)

$$\Rightarrow B = \phi$$

$$\bar{Y} \cap B = \phi$$

a contradiction, $B \neq \phi$

similarly

$$Y \subset B \Rightarrow A \neq \phi$$

Again we get contradiction of our hypothesis so, $\bar{Y}$ is connected.

Hence, the required result follow.

Pg 184

Component definition

(22) (18)

Th^m (1) - Every Component of a topological space are closed.

proof -

Let A be component of a topological space (X, T)

Then we have to show that A is closed for this we have to show.

$$\bar{A} = A$$

$\because A$ is component of X then

(i) A is connected.

(ii) BCX , B is connected $\Rightarrow A \subset B \subset A$

A is connected $\Rightarrow \bar{A}$ is connected.

$$\Rightarrow \bar{A} \subset A$$

$$A \subset \bar{A}$$

$\therefore \bar{A} = A \Rightarrow A$ is closed.

Pg. 188

7.6 Locally connected spaces. —

A topological space (X, τ) is called as locally connected at $x \in X$ iff every open nbd of x consists of a connected open nbd of x i.e. iff the class of all connected open nbds of x forms a local base at x .

Ex - Every discrete space is locally connected. $x \in X, \{x\}$ is connected & open.

Pg. 189

Th^m - Every component of a locally connected space (X, τ) is open.

proof - Let C be a component of X , then any $x \in C$ and A is locally connected.

$\Rightarrow x \in$ at least one connected open set G_x .

$\therefore x \in C$ & C is component of X

$\Rightarrow x \in G_x \subset C$

$\Rightarrow A = \bigcup \{G_x : x \in A\}$

Thus being union of open sets, A is open.

every open nbd of x must contain $\{x\}$

Pg. 189

Th^m-3 Every open subspace of a locally connected space is locally connected.

Proof - Let (Y, U) be an open subspace of a locally connected space (X, T)
 i.e. $Y \in T$ and $Y \subset X$

Let $y \in Y$ be arbitrary then $y \in X$.

To prove that Y is locally connected, we have to prove that Y is locally connected at y .

Let G be a U -open nbd of y , then $G \in T$
 for $A \in U \quad \forall A \in T$ as $Y \in T$ &
 $A \subset Y$

Now G is T -nbd of $y \in X$

Also X is locally connected.

Hence $\exists$ connected T -open nbd H of y
 s.t. $y \in H \subset G$

A is T -connected $\Leftrightarrow A$ is U -connected
 if $A \subset Y$.

Hence Y is locally connected at y .

Pg - 211

(19)

(23)

5.6

Sequitally Compact (spaces)

A topological space (X, T) is said to be sequentially compact iff every sequence $\{x_n\}$ in X has convergent subsequence.

(19) (22)

locally Compact - A topological space (X, T) is said to be locally compact iff every point in X has at least one nbd whose closure is compact.

213Th^m - 2 (19) ✓

Any open subspace of a locally compact space is locally compact.
 proof - let (X, T) be a compact space.
 and let (Y, U) be a open subspace of (X, T)

so that Y is open in X .

To prove Y is locally compact.

let $x \in Y \subset X$ be arbitrary
 and G be U -nbd of x in Y ,

Then $x \in X$, $G \subset Y$

X is locally compact
So, $\exists$ every point of X has atleast
one nbd whose closure is compact.

$x \in X$ is locally compact at x .
 G is a U -nbd of x in $Y \Rightarrow \exists G_1 \in U$
s.t. $x \in G_1 \subset G$

$\Rightarrow G_1 \in T$ s.t. $x \in G_1 \subset G$

for Y is open in X .

$\Rightarrow G$ is a T -nbd of x in X .

Also X is locally compact

$\Rightarrow \exists$ a compact set $A \subset X$

s.t.

$$x \in A^\circ \subset A \subset G$$

but $G \subset Y$

$$\Rightarrow x \in A^\circ \subset A \subset G \subset Y$$

Thus, $A \subset Y$, A is U -compact.

for A is T -compact $\Rightarrow U$ is compact.

(ii) G is nbd of x in Y s.t.

$$x \in A^\circ \subset A \subset G$$

hence Y is locally compact.

pg-213.

(22) (18) (23)

Th^m 18.22 Every closed subspace of a locally compact space is locally compact.

proof - Suppose (Y, U) be closed subspace of locally compact space (X, T) i.e. Y is T -closed set.

Let $y \in Y \subset X$ be arbitrary.

To prove that Y is closed locally compact we have to prove that Y is locally compact at y .

X is locally compact $\Rightarrow X$ is locally compact

$\Rightarrow \exists T$ -open nbd N of x s.t. $\bar{N}$ is T -c.

$\Rightarrow N \cap Y$ is U -open nbd of y .

$$N \cap Y \subset N \Rightarrow \overline{N \cap Y} \subset \bar{N}$$

Thus $\overline{N \cap Y}$ is a closed subset of a compact set $\bar{N}$.

Hence $\overline{N \cap Y}$ is compact.

Y is T -closed.

T -closed closure of $NAY = U$ -closure of NAY .

Thus NAY is U -Open nbd of y .
 s.t. $\overline{NAY}$ is compact.

Hence Y is locally compact at y .

pg. 216 # (21) ✓

Th^m-8 A continuous image of a sequentially compact set is sequentially compact.

proof -

Let $f: (X, T_1) \rightarrow (Y, T_2)$ be a continuous function.

and let A be a sequentially compact set of X .

To prove that $f(A)$ is sequentially compact.

Let $\{y_1, y_2, \dots\}$ be a sequence in $f(A)$
 $\exists x_1, x_2, \dots \in A$

s.t.

$$f(x_1) = y_1, \quad f(x_2) = y_2, \dots$$

since, A is sequentially compact $\{x_1, x_2, \dots\}$ contains a subsequence $\{x_{k_1}, x_{k_2}, \dots\}$

which converges to a point $x \in A$

Here $\{f(x_{k_1}), f(x_{k_2}), f(x_{k_3}) \dots\}$
converges to $f(x) \in f(A)$.

Hence, $f(A)$ is sequentially compact.

Pg 216

Theorem 10. Show that compactness is a topological invariant.

proof - Let (X, T) be a topological space
and let (Y, P) be a homeomorphism
space.

and f is the corresponding homeomorphism
s.t.

$$f: X \rightarrow Y$$

Then (i) f is one-one

(ii) f is continuous.

(iii) f^{-1} is continuous.

Now if (X, T) be compact space and A
is compact subset of X .

then $f(A)$ is compact.

i.e., compact $A \subset X$

$\Rightarrow f(A) \subset Y$ is compact.

Thm- (X, T) is disconnected $\Leftrightarrow \exists$ non-empty proper subset of X which is both open and closed.

$\Rightarrow A \subset X$, both open and closed.
proper $\neq \phi$

T.S. X is disconnected.

Let $B = A' \Rightarrow B \neq \phi$

Also $A \cap B = \phi$ & $A \cup B = X$

B is both open and closed.

$\Rightarrow \bar{A} = A$ & $\bar{B} = B$

$\Rightarrow A \cap \bar{B} = \phi$ & $\bar{A} \cap B = \phi$

$X =$ union of two separated set $\Rightarrow X$ is disconnected.

$\Leftarrow$ Let X be a disconnected $\Rightarrow$

$\exists A$ & B non empty subset of X

s.t. $A \cap \bar{B} = \phi$ & $\bar{A} \cap B = \phi$

$A \cup B = X$

Since $A \subset \bar{A}$, $\bar{A} \cap B = \phi \Rightarrow A \cap B = \phi$

$$(A = B', B = A')$$

A & B are proper subset of X .

$$A \cup B = X \Rightarrow A \cup \bar{B} = X$$

$$\hookrightarrow \bar{A} \cap B = \emptyset \Rightarrow A = (\bar{B})'$$

$$A \cap B = \emptyset$$

$$\text{similarly } B = (\bar{A})'$$

$\Rightarrow A$ & B are open.

$$A = B' \Rightarrow A \text{ is closed.}$$

similarly B is closed.

✓ Th.^m - F be connected subset of X . If F is subset of X s.t. $E \subset F \subset \bar{E} \Rightarrow F$ is connected. In particular $\bar{E}$ is connected.

✓ The closure of connected set is connected.
proof -

Let F is disconnected
then $\exists$ non-empty set A & B

$$\text{s.t. } A \cap \bar{B} = \emptyset, \bar{A} \cap B = \emptyset \text{ \& } A \cup B = F$$

since

$$ECF = A \cup B$$

$$\Rightarrow ECA \text{ or } ECB$$

$$\text{let } ECA \Rightarrow \bar{E} \subset \bar{A} \Rightarrow \bar{E} \cap B \subset \bar{A} \cap B = \emptyset$$

$$\bar{E} \cap B = \emptyset \quad \text{--- (1)}$$

$$A \cup B = F, \quad F \subset \bar{E}$$

$$\Rightarrow B \subset F \subset \bar{E}$$

$$B \subset F$$

$$\Rightarrow B \cap \bar{E} = B \quad \text{--- (2)}$$

$$\Rightarrow B = \emptyset$$

This is contradiction.

F is connected thus $\bar{E}$ is also connected.

Thm^m - ✓ Subset E of $\mathbb{R}$ is connected $\Leftrightarrow E$ is an interval.

$\Rightarrow$ let E is connected.

if $E = \emptyset$ then Nothing to prove.

let E contains more than one point

let E is not an interval.

$$-\infty \quad \underbrace{\quad \quad \quad}_{\substack{a \quad p \quad b}} \quad \infty$$

$\Rightarrow \exists a, p, b$ with $a < p < b$

s.t. $a, b \in E, p \notin E$

$$\text{let } A = (p, \infty) \cap E$$

$$A \neq \emptyset \\ b \in A$$

$$\text{and } B = (-\infty, p) \cap E$$

$$B \neq \emptyset \\ a \in B$$

$$\overline{A} \cap B = [(\overline{(p, \infty) \cap E}) \cap ((-\infty, p) \cap E)]$$

$$\subseteq (\overline{p}, \infty) \cap \overline{E} \cap ((-\infty, p) \cap E)$$

$$= [p, \infty) \cap \overline{E} \cap ((-\infty, p) \cap E)$$

$$= [p, \infty) \cap (-\infty, p) \cap E \quad (\because E \subseteq \mathbb{R})$$

$$= \emptyset$$

$$\overline{A} \cap B = \emptyset$$

$$\text{similarly } A \cap \overline{B} = \emptyset$$

$$A \cup B = ((p, \infty) \cap E) \cup ((-\infty, p) \cap E)$$

$$= [(-\infty, p) \cup (p, \infty)] \cap E$$

$$= [\mathbb{R} - \{p\}] \cap E$$

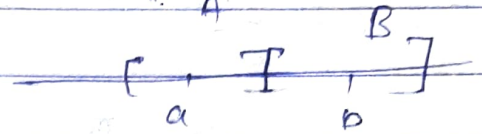
$$= E$$

E is disconnected.

Contradiction.

~~$\nexists$~~ ~~$\nexists$~~ $\rightarrow E$ is interval. A

Conversely :-



E be an interval.

If possible E be disconnected.

$\Rightarrow \exists$ non-empty disjoint both closed.
in E s.t. $E = A \cup B$

Choose $a \in A$, $b \in B$

since $A \cap B = \emptyset \Rightarrow a \neq b$

$\Rightarrow$ either $a < b$ or $b < a$

without loss of generality $a < b$

since E is an interval & $a, b \in E$

$\Rightarrow [a, b] \subset E = A \cup B$

let $u = \sup([a, b] \cap A)$

$\Rightarrow a \leq u \leq b$

To each $\epsilon > 0 \exists v \in [a, b] \cap A$

s.t. $u - \epsilon < v \leq u$.

$\Rightarrow$ Every nhd of u contains point of $[a, b] \cap A \Rightarrow u \in A$ or $u \in D(A)$

but A is closed $\Rightarrow u \in A$

$\Rightarrow u \notin B$

but definition of u shows that-

$u + \epsilon \in B \quad \forall \quad \epsilon > 0 \quad \text{s.t.} \quad u + \epsilon \leq b$

$\Rightarrow$ Every nhd of u contains a point B distinct from $u \Rightarrow$

$u \in D(B)$ but B is closed.

$\Rightarrow u \in B$

Contradiction-

B is connected.

Th^m - Continuous image of a connected space is connected.

proof -

let $f: X \rightarrow Y$ be continuous mapping of a connected space X into an arbitrary topological space Y .

T.S. $f(X)$ is connected subspace of Y
let $f(X)$ be disconnected,

$\Rightarrow \exists G_1$ & G_2 both open in Y s.t.

$$G_1 \cap f(X) \neq \emptyset \quad \& \quad G_2 \cap f(X) \neq \emptyset$$

$$\text{and } (G_1 \cap f(X)) \cap (G_2 \cap f(X)) = \emptyset \quad \text{--- (1)}$$

$$(G_1 \cap f(X)) \cup (G_2 \cap f(X)) = f(X) \quad \text{--- (2)}$$

$$\Rightarrow \emptyset = f^{-1}(\emptyset) = f^{-1}[(G_1 \cap f(X)) \cap (G_2 \cap f(X))]$$

$$= f^{-1}[(G_1 \cap G_2) \cap f(X)]$$

$$= f^{-1}(G_1) \cap f^{-1}(G_2) \cap f^{-1}f(X)$$

$$= f^{-1}(G_1) \cap f^{-1}(G_2) \cap X$$

$$= f^{-1}(G_1) \cap f^{-1}(G_2)$$

$$X = f^{-1}(X) = f^{-1}[(G_1 \cap f(X)) \cup (G_2 \cap f(X))]$$

$$= f^{-1}[(G_1 \cup G_2) \cap f(X)]$$

$$= f^{-1}[G_1 \cup f^{-1}(G_2) \cap f^{-1}f(X)]$$

$$= [f^{-1}(G_1) \cup f^{-1}(G_2)] \cap X$$

$$= f^{-1}(G_1) \cup f^{-1}(G_2)$$

f is continuous $\Rightarrow f^{-1}(G_1)$ & $f^{-1}(G_2)$ both are open subsets of X

$\Rightarrow X$ is disconnected.

contradiction.

$\Rightarrow f(X)$ is connected.

Components -

If (X, τ) is not itself connected then it can be decomposed into a disjoint class of maximal connected subspaces.

✓ Totally Disconnected Space.

$\Rightarrow (X, \tau)$ is totally disconnected iff for each pair of distinct points $x, y \in X$, $\exists$ disconnection $A \cup B \supset X$ s.t. $x \in A$ & $y \in B$. Totally connected space $\Rightarrow$ Hausdorff

Ex. $\Rightarrow$ Discrete space is totally ^{dis}connected.
 $x, y \in X$, $A = \{x\}$, $B = X - \{x\}$

A & B both $\neq \emptyset$ open sets s.t.

$$X = A \cup B, \quad x \in A, y \in B$$

Ex. $(\mathbb{R}, \tau)$ is totally disconnected

$$x < y \quad A = (-\infty, y), \quad B = [y, \infty)$$

$A \cup B = \mathbb{R}$, A & $B \neq \emptyset$ disjoint open sets.

$$x \in A, y \in B$$

Ex. $(\mathbb{Q}, \cup)$ is totally disconnected.

$$x < y \Rightarrow \exists p \in \mathbb{Q}^c \text{ s.t.}$$

$$x < p < y$$

$$A = \mathbb{Q} \cap (-\infty, p)$$

$$B = \mathbb{Q} \cap (p, \infty)$$

$$A \cup B = \mathbb{Q}, \quad x \in A, y \in B$$

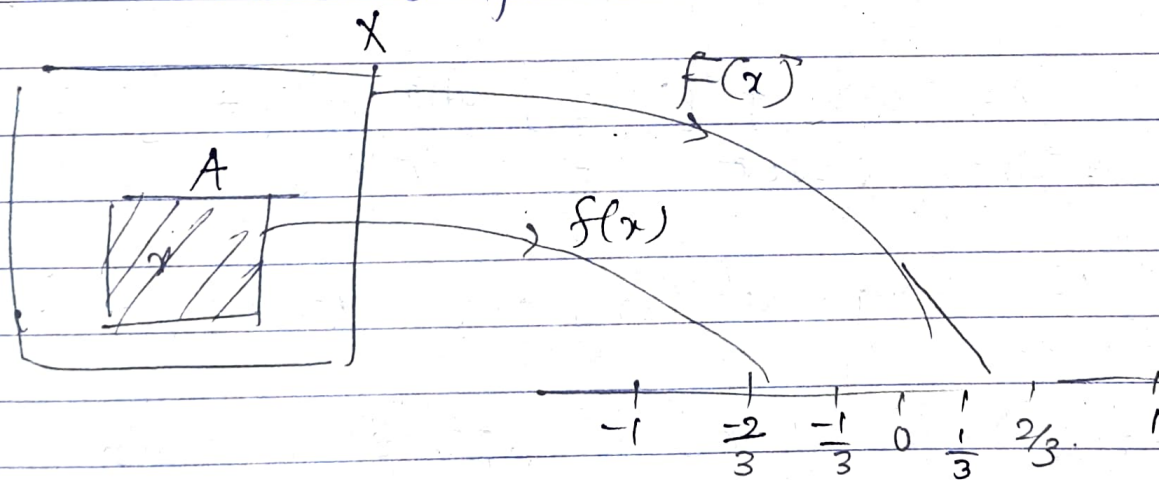
similarly set $\mathbb{Q} \cap \mathbb{Q}^c$ is totally connected.

* Tietz Extension Theorem:—

Statement $\Rightarrow$ Let A be a closed subset of a normal space X and suppose $f: A \rightarrow [-1, 1]$ is a continuous function, then $\exists$ a continuous function $F: X \rightarrow [-1, 1]$ such that

$$F(x) = f(x) \quad \forall x \in A$$

proof $\Rightarrow$ Let A be a closed subset of a normal space X and suppose $f: A \rightarrow [-1, 1]$ is a continuous function.



let

$$B_1 = \left\{ x \in A \mid -1 \leq f(x) \leq -\frac{1}{3} \right\}$$

$$\therefore B_1 = \left\{ x \in A \mid f(x) \in \left[-1, -\frac{1}{3} \right] \right\}$$

$$B_1 = \{x \in f^{-1}[-1, -\frac{1}{3}]\}$$

let

$$C_1 = \{x \in A \mid \frac{1}{3} \leq f(x) \leq 1\}$$

$$C_2 = \{x \in A \mid f(x) \in [\frac{1}{3}, 1]\}$$

$$C_1 = \{x \in f^{-1}[\frac{1}{3}, 1]\}$$

since $[-1, -\frac{1}{3}]$ and $[\frac{1}{3}, 1]$ are disjoint

closed set and f is continuous.

Therefore $f^{-1}[-1, -\frac{1}{3}]$ and $f^{-1}[\frac{1}{3}, 1]$ are disjoint closed set in X .

By general Urysohn's lemma,

"If X is a normal space then for any two disjoint closed set A and B of X $\exists$ a continuous function $f: X \rightarrow [a, b]$ such that $f(A) = \{a\}$

and $f(B) = \{b\}$

$\exists$ a continuous function

$f_1 : X \rightarrow [-\frac{1}{3}, \frac{1}{3}]$ such that

$$f_1(B_1) = \{-\frac{1}{3}\}, \quad f_1(C_1) = \{\frac{1}{3}\}$$

we observe that

① f_1 is continuous function

② $|f_1(x)| < \frac{1}{3} = \frac{2^{1-1}}{3} = \frac{1}{3} \quad \forall x \in X$

③ $|f(x) - f_1(x)| \leq \frac{2}{3} \quad \forall x \in A$

proof of ③

$$|f(x) - f_1(x)| = |f(x) - f(x)|$$

$$\because x \in B_1$$

$$\therefore x \in f^{-1}[-1, -\frac{1}{3}]$$

$$\Rightarrow f(x) \in [-1, -\frac{1}{3}]$$

$$\text{now } \max [f(x) - f_1(x)] = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\therefore \frac{2}{3} = \max [f(x) - f_1(x)]$$

$$\Rightarrow \frac{2}{3} \geq f(x) - f_1(x) \quad \text{--- (1)}$$

$$\text{now } \min [f(x) - f_1(x)] = -1 + \frac{1}{3} = -\frac{2}{3}$$

$$\Rightarrow -\frac{2}{3} = \min [f(x) - f_1(x)]$$

$$\Rightarrow -\frac{2}{3} \leq f(x) - f_1(x) \quad \text{--- (B)}$$

from eqⁿ (A) & (B)

$$-\frac{2}{3} \leq f(x) - f_1(x) \leq \frac{2}{3}, \quad \forall x \in A$$

$$\Rightarrow |f(x) - f_1(x)| \leq \frac{2}{3}, \quad \forall x \in A$$

$$\Rightarrow f(x) - f_1(x) \in \left[-\frac{2}{3}, \frac{2}{3}\right]$$

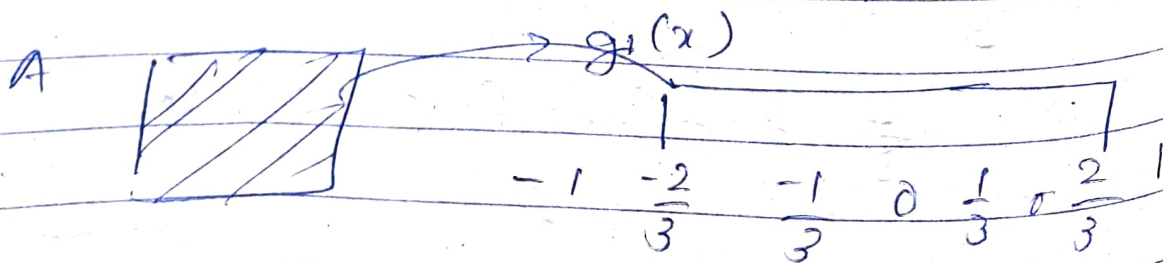
Again Define a function given by

$$g_1: A \rightarrow \left[-\frac{2}{3}, \frac{2}{3}\right] \text{ such that}$$

$$g_1(x) \equiv f(x) - f_1(x)$$

Then

(1) $g_1(x)$ is continuous



$$(i) |g_1(x)| \leq \frac{2}{3}$$

$$\text{let } B_2 = \{x \in A \mid g_1(x) \leq -\frac{2}{9}\}$$

$$\text{and } C_2 = \{x \in A \mid g_1(x) \geq \frac{2}{9}\}$$

$$\text{Then } B_2 = g_1^{-1} \left\{ -\frac{2}{3}, -\frac{2}{9} \right\}$$

$$\text{and } C_2 = g_1^{-1} \left\{ \frac{2}{3}, \frac{2}{9} \right\}$$

Since $g_1(x)$ is continuous

$\therefore B_2$ and C_2 both are closed set

$$B_2 \cap C_2 = g_1^{-1} \left[\left\{ -\frac{2}{3}, -\frac{2}{9} \right\} \cap \left\{ \frac{2}{9}, \frac{2}{3} \right\} \right]$$

$$B_2 \cap C_2 = g_1^{-1} \{ \emptyset \} = \emptyset$$

Therefore B_2 and C_2 are Disjoint subset set.

By general Urysohn's Lemma,

$\exists$ a continuous function,

$$f_2 : X \rightarrow \left[-\frac{2}{9}, \frac{2}{9} \right] \text{ such that}$$

$$f_2(B_2) = -\frac{2}{9} \quad , \quad f_2(C_2) = \frac{2}{9}$$

$$\text{And } -\frac{2}{9} \leq f_2(x) \leq \frac{2}{9}$$

then

① $f_2(x)$ is continuous.

$$② |f_2(x)| \leq \frac{2}{9} = \frac{2^1}{3^2} \quad \forall x \in X$$

$$③ |g_1(x) - f_2(x)| \leq \frac{4}{9} = \frac{2^2}{3^2} \quad \forall x \in A$$

In general case, we get-

$$|f(x) - \sum_{i=1}^n f_i^o(x)| \leq \frac{2^n}{3^n} \quad \text{--- ①}$$

f_i^o is continuous

$$|f_n(x)| \leq \frac{2^{n-1}}{3^n} \quad \forall x \in X$$

$$|f(x) - \sum_{i=1}^n f_i^o(x)| \leq \frac{2^n}{3^n} \quad \forall n \in \mathbb{N} \text{ and } x \in A$$

$$\text{If } M_n = \frac{2^{n-1}}{3^n}$$

$$\text{Then } \sum M_n = \sum \frac{2^{n-1}}{3^n} \quad \text{is}$$

convergent (because it is geometric series with common ratio less than 1)

then $\sum M_n$ is uniformly convergent say $(F(x))$

$$\text{and } \sum f_n(x) \leq \sum \frac{2^{n-1}}{3^n}$$

By Weierstrass M_n -test

$\sum f_n(x)$ is also uniform convergent

$$\therefore \sum f_n(x) \rightarrow F(x)$$

$\therefore f_n(x)$ is continuous.

$\therefore F(x)$ is also continuous.

Now by eq (i) we get.

$$\left| f(x) - \sum_{i=1}^n f_i(x) \right| \leq \frac{2^n}{3^n} = \varepsilon (\text{say}) \quad \forall x \in A$$

$$\Rightarrow \sum f_i(x) \rightarrow f(x)$$

i.e. $\sum f_n(x)$ convergence to $f(x)$.

but $\sum f_n(x)$ uniformly converge to $F(x)$. $\forall x \in X$

So by uniqueness of the limit,

$$\text{we get } F(x) = f(x) \quad \forall x \in A$$

This completes the proof.

★ Completely Regular space / Tychonoff space:

A topological space X is completely regular $\Leftrightarrow$ for every closed sets subset F of X and every point $x \in X - F$
 $\exists$ continuous map $f: X \xrightarrow{\text{into}} [0,1]$
 s.t. $f(x) = 0$ & $f(F) = 1$.

Tychonoff space / $T_{3\frac{1}{2}}$ space:-

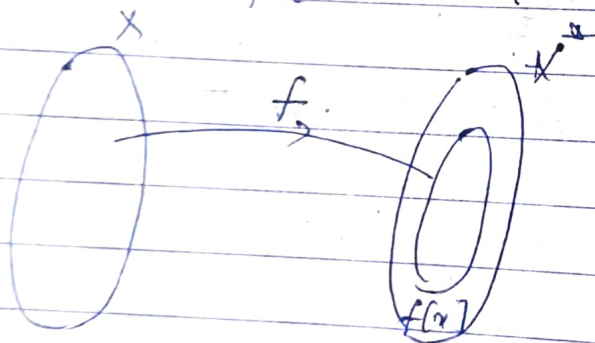
Completely Regular + T_1 space.

Compactification $\Rightarrow$ A compact space (X^*, τ) is a compactification of (X, τ) iff $\exists$ a mapping $f: X \rightarrow X^*$

f is homeomorphism

$$f[X] \subseteq X^*$$

$$\overline{f[X]} = X^*$$



One point compactification $\Rightarrow$

$$X^* = X \cup \{\infty\}$$

$$C: X \rightarrow X^*$$

1. X is Hausdorff, non compact, locally compact.
2. C is continuous, open
3. X^* is compact.

4. X^* is Hausdorff $\Leftrightarrow X$ is Hausdorff and locally compact.

∞ - any point not in X .

$$\hookrightarrow X: (X, \tau), (X^*, \tau^*)$$

$$\tau^* = \tau \cup \{(x|c) \cup \{\infty\} : c \text{ is compact and closed in } X\}$$

Theorem $\Rightarrow$

Statement - let (X, τ) be a completely regular space. There exist a compactification Y of X having the property that every bounded continuous map

$$f: X \rightarrow \mathbb{R}$$

$f: X \rightarrow \mathbb{R}$ extends uniquely to a continuous map of Y into $\mathbb{R}$.

or